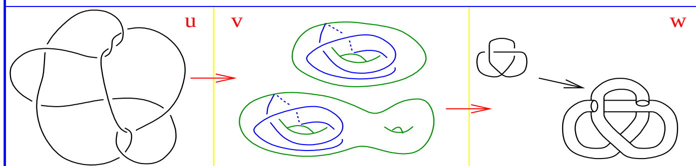
Day 1 − u, v, w: topology and philosophy

Dror Bar−Natan, Goettingen, April 2010

Plans and Dreams

$$\begin{pmatrix}\text{arbitrary algebraic}\\ \text{structure}\end{pmatrix} \xrightarrow[\text{machine}]{\text{projectivization}} \begin{pmatrix}\text{a problem in}\\ \text{graded algebra}\end{pmatrix}$$

- Feed knot-things, get Lie algebra things.
- Feed u-knots, get Drinfel'd associators.
- Feed w-knots, get Kashiware-Vergne-Alekseev-Torossian.
- Dream: Feed v-knots, get Etingof-Kazhdan.
- Dream: Knowing the question whose answer is 42, or E-K, will be useful to algebra and topology.

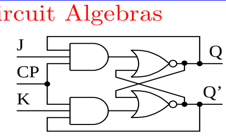


u-Knots

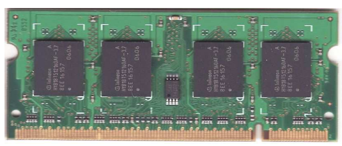
(PA :=Planar Algebra)

$$\begin{Bmatrix}\text{knots}\\ \text{\&links}\end{Bmatrix} = PA\left\langle \;\middle|\; \text{R123: } \right\rangle_{0 \text{ legs}}$$

Circuit Algebras



A J-K Flip Flop



Infineon HYS64T64020HDL-3.7-A 512MB RAM

v-Knots

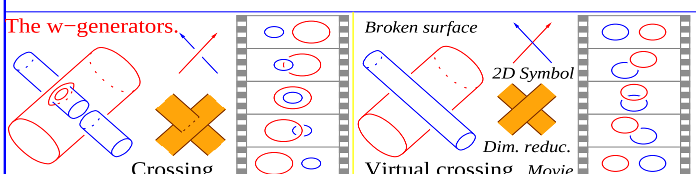
(CA :=Circuit Algebra)

$$\begin{Bmatrix}\text{v-knots}\\ \text{\& links}\end{Bmatrix} = CA\left\langle \;\middle|\; \text{R23: } \right\rangle_{0 \text{ legs}}$$

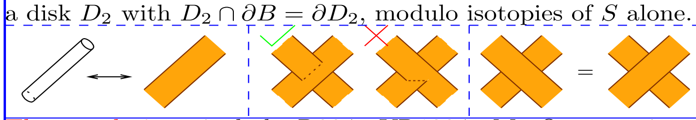
$$= PA\left\langle \;\middle|\; \text{VR123: } ;\quad \text{R23; D: } \right\rangle_{0 \text{ legs}}$$

w-Tangles

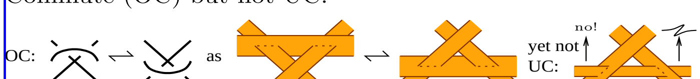
$$\{\text{w-Tangles}\} = \text{v-Tangles} \Big/ OC :$$



A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

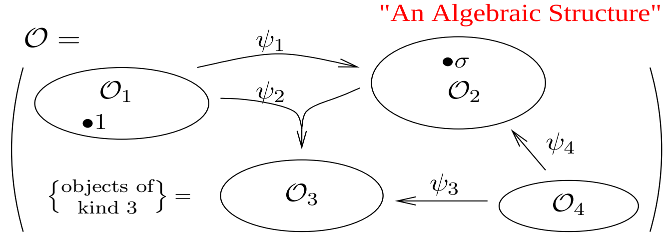


The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC:



Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{array}{ccccccccccc} \text{ops} \circlearrowleft \mathcal{K} & = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 & \supset & \dots \\ \Downarrow & & & & & & \downarrow Z & & & & \end{array}$$

$$\text{ops} \circlearrowleft \operatorname{gr}\mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

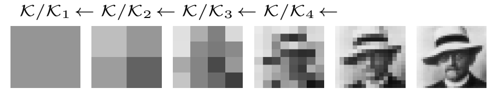
An expansion is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

Just for fun.

$$\mathcal{K} = \left\{ \right\} = \begin{pmatrix}\text{The set of all}\\ \text{b/w 2D projec-}\\ \text{tions of reality}\end{pmatrix}$$

$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow$ ...

Crop Rotate Adjoin



...

An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin

$$\underset{\mathbb{R}}{\overset{\|}{\mathcal{K}/\mathcal{K}_1}} \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \underset{\ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)}{\overset{\|}{\mathcal{K}_3/\mathcal{K}_4}} \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \cdots$$

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products").

Examples. 1. The projectivization of a group is a graded associative algebra. 2. Quandle: a set Q with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

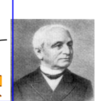
$\operatorname{proj} Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

Our case(s).

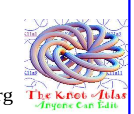
$$\mathcal{K} \xrightarrow[\substack{\text{solving finitely many}\\ \text{equations in finitely}\\ \text{many unknowns}}]{Z:\ \text{high algebra}} \begin{matrix}\mathcal{A} :=\\ \operatorname{proj}\mathcal{K}\end{matrix} \xrightarrow[\substack{\text{low algebra: pic-}\\ \text{tures represent}\\ \text{formulas}}]{\substack{\text{given a "Lie"}\\ \text{algebra } \mathfrak{g}}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

$\mathcal{K}$ is knot theory or topology; $\operatorname{proj}\mathcal{K} = \bigoplus \mathcal{I}^m/\mathcal{I}^{m+1}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org



Video and more at http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/