

Balloons and Hoops and their Universal Finite-Type Invariant, BF Theory, and an Ultimate Alexander Invariant

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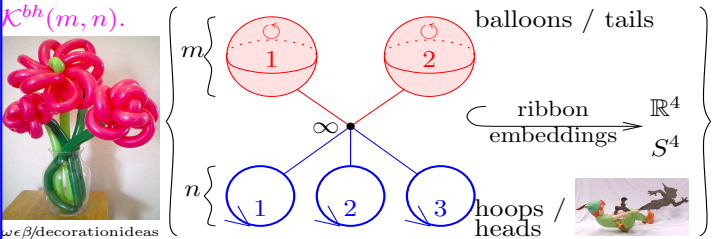
$\omega \in \beta := \text{http://www.math.toronto.edu/~drorbn/Talks/Oxford-130121}$



Scheme. • Balloons and hoops in \mathbb{R}^4 , algebraic structure and relations with 3D.

• An ansatz for a “homomorphic” invariant: computable, related to finite-type and to BF.

• Reduction to an “ultimate Alexander invariant”.



Examples.

ϵ_x :

ϵ_u :

ρ_{ux}^+ :

ρ_{ux}^- :

I mean business!

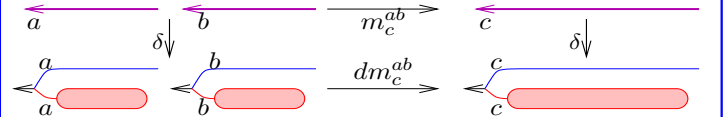
$T_0 = \text{Rm}[3, a] \text{Rp}[2, 2] \text{Rp}[1, 4]$
 $S = T_0 // \text{dm}[2, 1, 1] // \text{dm}[4, b, b] //$
 $\text{dm}[1, a, a] // \text{dm}[3, a, a]$
 $S[[S]] / \langle w_{CW} \Rightarrow (\text{Deg}[w] + 1) \cdot w, w_{CW} \Rightarrow \text{Deg}[w] \cdot w \rangle$

$\mu[\text{CWG}[-[a], -2[\text{ab}], -3[\text{abb}], -3[\text{abb}], -4[\text{aabb}] - 42[\text{aabb}] - 60[\text{abab}] - 4[\text{abbb}], -5[\text{aaab}] - 110[\text{aaabb}] - 180[\text{aabb}] - 110[\text{ababb}] - 180[\text{ababb}] - 5[\text{abbbb}], \text{h}[b] \text{LS}[2(a), 0, -24(\text{aab}), -60(\text{aabb}) + 60(\text{aabb}), -120(\text{aaab}) + 900(\text{aabb}) + 360(\text{aabb}) - 120(\text{aaabb})] + \text{h}[a] \text{LS}[-2(a) + 2(b), 9(\text{ab}), 26(\text{aab}) - 26(\text{abb}), 60(\text{aaab}) - 255(\text{aabb}) - 60(\text{abbb}), 119(\text{aaab}) - 1504(\text{aaabb}) - 118(\text{aabb}) + 1504(\text{aabb}) - 1386(\text{ababb}) - 119(\text{abbbb})]$

Meta-associativity.

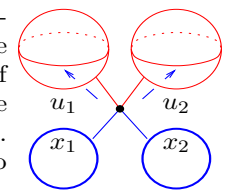


Tangle concatenations $\rightarrow \pi_1 \times \pi_2$.



Thus we seek homomorphic invariants of \mathcal{K}^{bh} !

Invariant #0. With Π_1 denoting “honest π_1 ”, map $\gamma \in \mathcal{K}^{bh}(m, n)$ to the triple $(\Pi_1(\gamma^c), (u_i), (x_j))$, where the meridian of the balls u_i normally generate Π_1 , and the “longitudes” x_j are some elements of Π_1 . $*$ acts like $*$, tm acts by “merging” two meridians/generators, hm acts by multiplying two longitudes, and tha^{ux} acts by “conjugating a meridian by a longitude”:



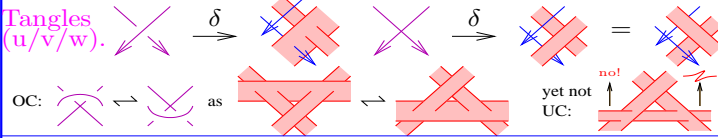
Not computable! (but nearly)

$(\Pi, (u, \dots), (x, \dots)) \mapsto (\Pi \langle \bar{u} \rangle / (u = x\bar{u}x^{-1}), (\bar{u}, \dots), (x, \dots))$

Failure #0. Can we write the x 's as free words in the u 's?

If $x = uv$, compute $x // tha^{ux}$:

$$x = uv \rightarrow \bar{u}v = u^xv = u^{\bar{u}v}v = u^{u^xv}v = u^{u^{ux}v}v = \dots$$

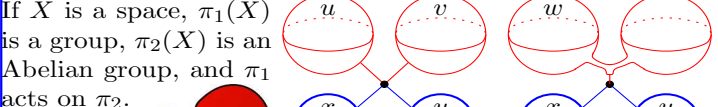


• δ injects u-Knots into \mathcal{K}^{bh} (likely u-tangles too).

• δ maps v/w-tangles map to \mathcal{K}^{bh} ; the kernel contains Reidemeister moves and the “overcrossings commute” relation, and **conjecturally**, that's all. Allowing punctures and cuts, δ is onto.

Operations
Punctures & Cuts

Meta-Group-Action. K : $K // tm_w^{uv}$:



If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2 .

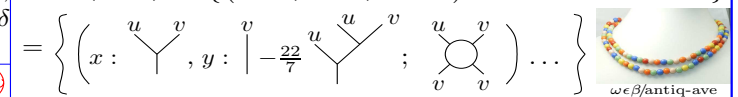
“MGA”

(“//” is newspeak for “apply an operator” and for “composition left to right”)

Properties.

- Associativities: $m_a^{ab} // m_a^{ac} = m_b^{bc} // m_a^{ab}$, for $m = tm, hm$.
- Action axiom t : $tm_w^{uv} // tha^{wx} = tha^{ux} // tha^{vx} // tm_w^{uv}$,
- Action axiom h : $hm_z^{xy} // tha^{uz} = tha^{ux} // tha^{uy} // hm_z^{xy}$.
- SD Product: $dm_c^{ab} := tha^{ab} // tm_c^{ab} // hm_c^{ab}$ is associative.

The Meta-Group-Action M . Let T be a set of “tail labels” (“balloon colours”), and H a set of “head labels” (“hoop colours”). Let $FL = FL(T)$ and $FA = FA(T)$ be the (completed graded) free Lie and free associative algebras on generators T and let $CW = CW(T)$ be the (completed graded) vector space of cyclic words on T , so there's $\text{tr} : FA \rightarrow CW$. Let $M(T, H) := \{(\bar{\lambda} = (x : \lambda_x)_{x \in H}; \omega) : \lambda_x \in FL, \omega \in CW\}$



Operations. Set $(\bar{\lambda}_1; \omega_1) * (\bar{\lambda}_2; \omega_2) := (\bar{\lambda}_1 \cup \bar{\lambda}_2; \omega_1 + \omega_2)$ and with $\mu = (\bar{\lambda}; \omega)$ define

$$tm_w^{uv} : \mu \mapsto \mu // (u, v \mapsto w),$$

$$hm_z^{xy} : \mu \mapsto ((\dots, \widehat{x : \lambda_x}, \widehat{y : \lambda_y}, \dots, z : \text{bch}(\lambda_x, \lambda_y)); \omega)$$

“stable apply”

$$tha^{ux} : \mu \mapsto \underbrace{\mu // (u \mapsto e^{\text{ad } \lambda_x}(\bar{u}))}_{\mu // CC_u^\lambda} // (\bar{u} \mapsto u) + (0; J_u(\lambda_x))$$

the “ J -spice”

A CC_u^λ example.

