Trees and Wheels and Balloons and Hoops

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ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/Zurich-130919

15 Minutes on Algebra

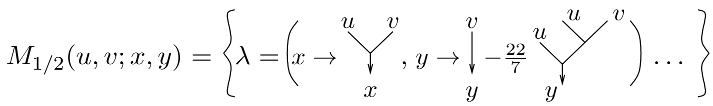
Let T be a finite set of "tail labels" and H a finite set of "head labels". Set

$$M_{1/2}(T;H) := FL(T)^H,$$

"H-labeled lists of elements of the degree-completed free Lie algebra generated by T".

$$FL(T) = \left\{2t_2 - \frac{1}{2}[t_1,[t_1,t_2]] + \dots\right\} \Big/ \left(\begin{matrix}\text{anti-symmetry}\\ \text{Jacobi}\end{matrix}\right)$$

... with the obvious bracket.



Operations $M_{1/2} \to M_{1/2}$. newspeak!

Tail Multiply tm^{uv}_w is $\lambda \mapsto \lambda /\!\!/ (u,v \to w)$, satisfies "meta-associativity", $tm^{uv}_u /\!\!/ tm^{uw}_u = tm^{vw}_v /\!\!/ tm^{uv}_u$.

Head Multiply hm^{xy}_z is $\lambda \mapsto (\lambda \setminus \{x,y\}) \cup (z \to \mathrm{bch}(\lambda_x,\lambda_y))$, where

$$\mathrm{bch}(\alpha,\beta) := \log(e^\alpha e^\beta) = \alpha + \beta + \frac{[\alpha,\beta]}{2} + \frac{[\alpha,[\alpha,\beta]]+[[\alpha,\beta],\beta]}{12} + \dots$$

satisfies $\mathrm{bch}(\mathrm{bch}(\alpha,\beta),\gamma) = \log(e^\alpha e^\beta e^\gamma) = \mathrm{bch}(\alpha,\mathrm{bch}(\beta,\gamma))$ and hence meta-associativity, $hm^{xy}_x /\!\!/ hm^{xz}_x = hm^{yz}_y /\!\!/ hm^{xy}_x$.

Tail by Head Action tha^{ux} is $\lambda \mapsto \lambda /\!\!/ RC^{\lambda_x}_u$, where $C^{-\gamma}_u\colon FL \to FL$ is the substitution $u \to e^{-\gamma}ue^{\gamma}$, or more precisely,

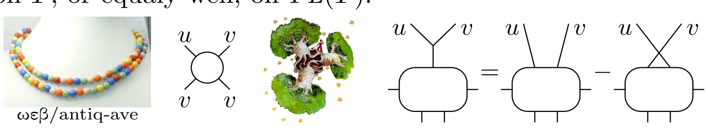
$$C^{-\gamma}_u\colon u \to e^{-\operatorname{ad}\gamma}(u) = u - [\gamma,u] + \frac{1}{2}[\gamma,[\gamma,u]] - \dots,$$

and $RC^\gamma_u = (C^{-\gamma}_u)^{-1}$. Then $C^{\mathrm{bch}(\alpha,\beta)}_u = C^{\alpha /\!\!/ RC^{-\beta}_u}_u /\!\!/ C^\beta_u$ hence $RC^{\mathrm{bch}(\alpha,\beta)}_u = RC^\alpha_u /\!\!/ RC^{\beta /\!\!/ RC^\alpha_u}_u$ hence "meta $u^{xy} = (u^x)^y$",

$$hm^{xy}_z /\!\!/ tha^{uz} = tha^{ux} /\!\!/ tha^{uy} /\!\!/ hm^{xy}_z,$$

and $tm^{uv}_w /\!\!/ C^{\gamma /\!\!/ tm^{uv}_w}_w = C^{\gamma /\!\!/ RC^{-\gamma}_v}_u /\!\!/ C^\gamma_v /\!\!/ tm^{uv}_w$ and hence "meta $(uv)^x = u^xv^x$", $tm^{uv}_w /\!\!/ tha^{wx} = tha^{ux} /\!\!/ tha^{vx} /\!\!/ tm^{uv}_w$.

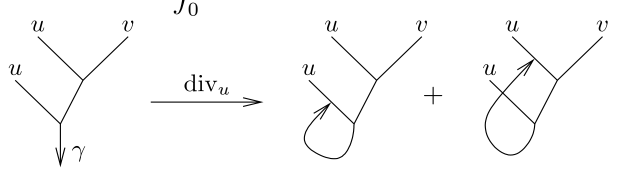
Wheels. Let $M(T;H) := M_{1/2}(T;H) \times CW(T)$, where $CW(T)$ is the (completed graded) vector space of cyclic words on T, or equaly well, on $FL(T)$:



Operations. On $M(T;H)$, define tm^{uv}_w and hm^{xy}_z as before, and tha^{ux} by adding some J-spice:

$$(\lambda;\omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) /\!\!/ RC^{\lambda_x}_u,$$

where $J_u(\gamma) := \int_0^1 ds\ \mathrm{div}_u(\gamma /\!\!/ RC^{s\gamma}_u) /\!\!/ C^{-s\gamma}_u$, and




Alekseev

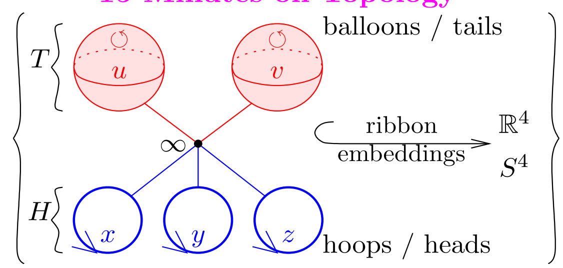

Torossian

Theorem Blue. All blue identities still hold.

Merge Operation. $(\lambda_1;\omega_1) * (\lambda_2;\omega_2) := (\lambda_1 \cup \lambda_2; \omega_1 + \omega_2)$.

15 Minutes on Topology

$\mathcal{K}^{bh}(T;H)$. "Ribbon-knotted balloons and hoops"



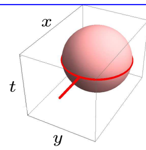
Examples.

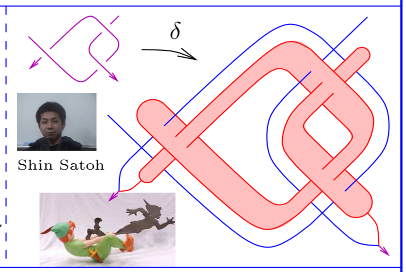
ϵ_x: x

ϵ_u: u

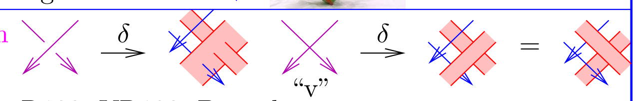
ρ^+_{ux}: u x ρ^-_{ux}: u x

"the generators"

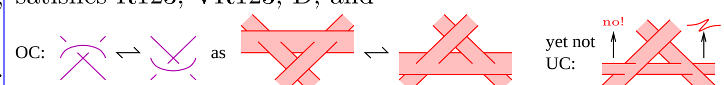




More on δ



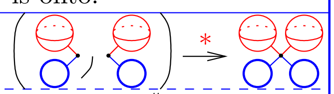
satisfies R123, VR123, D, and



- δ injects u-knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
- δ maps v-tangles to $\mathcal{K}^{bh}$; the kernel contains the above and conjecturally (Satoh), that's all.
- Allowing punctures and cuts, δ is onto.

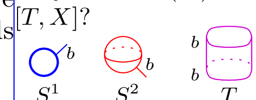
Operations Punctures & Cuts

Connected Sums.

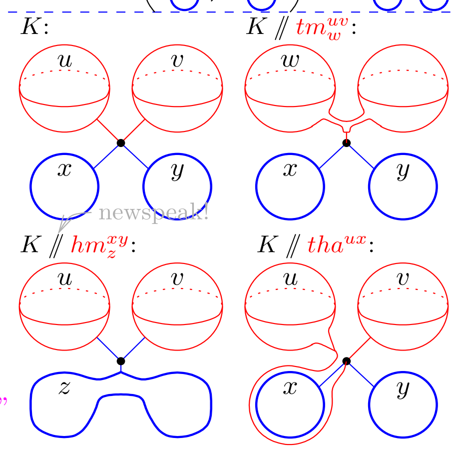


If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

Riddle. People often study $\pi_1(X) = [S^1,X]$ and $\pi_2(X) = [S^2,X]$. Why not $\pi_T(X) := [T,X]$?



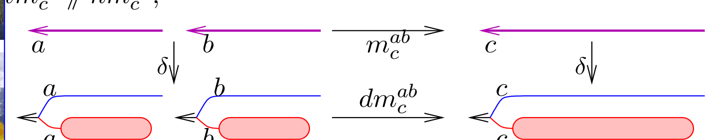
"Meta-Group-Action"



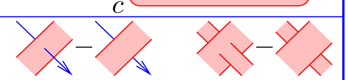
Properties.

- Associativities: $m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$, for $m = tm, hm$.
- "$(uv)^x = u^xv^x$": $tm^{uv}_w /\!\!/ tha^{wx} = tha^{ux} /\!\!/ tha^{vx} /\!\!/ tm^{uv}_w$,
- "$u^{(xy)} = (u^x)^y$": $hm^{xy}_z /\!\!/ tha^{uz} = tha^{ux} /\!\!/ tha^{uy} /\!\!/ hm^{xy}_z$.

Tangle concatenations → $\pi_1 \ltimes \pi_2$. With $dm^{ab}_c := tha^{ab} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$,



Finite type invariants make sense in the usual way, and "algebra" is (the primitive part of) "gr" of "topology".



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