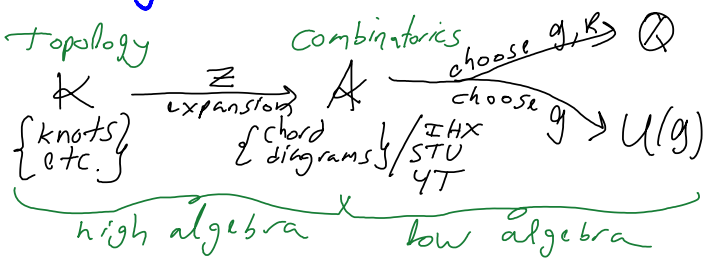
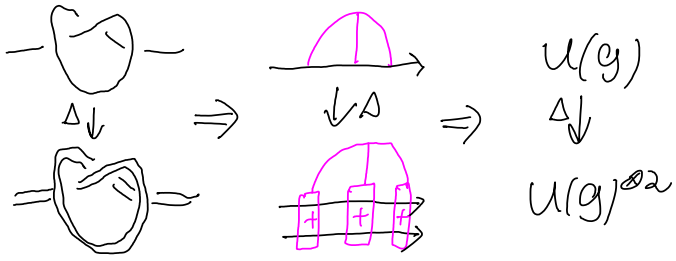
Low and High Algebra in the "u" Case

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

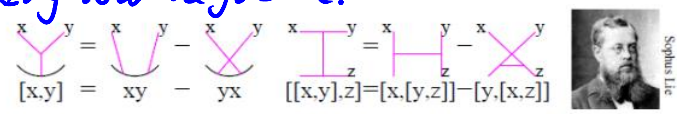
The big picture, "u" case.



What's Δ?



Very low algebra.



More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [a,b],c \rangle \qquad X_a v_\beta = \sum_\gamma r^\beta_{a\gamma} v_\gamma$$

and then

$$W_{\mathfrak{g},R}: \quad \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

Exercise. Find a fast method to find $W_{g,R}(D)$ when $g = gl_n$, $R = \mathbb{R}^n$. Is it related to the Conway polynomial?

A "Homomorphic Expansion" $Z: \mathcal{K} \to \mathcal{A}$

is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is **High Algebra.**

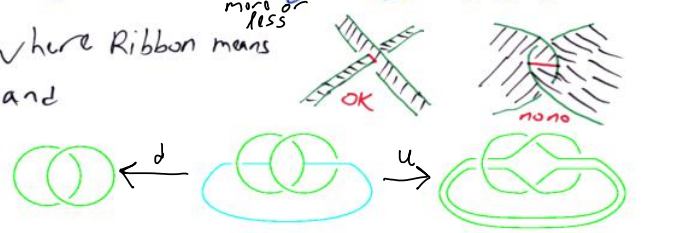
Universal Representation Theory.

Inspired by $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$, set $U(\mathfrak{g}) = \langle \text{words in } \mathfrak{g} \rangle / [x,y] = xy - yx$

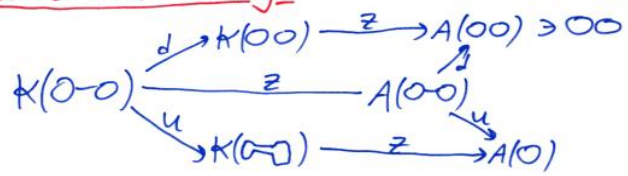
* Every rep of $\mathfrak{g}$ extends to $U(\mathfrak{g})$.
* $\exists\, \Delta: U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes 2}$ by "word splitting", as must be for $R \otimes R$.

Exercise. With $\mathfrak{g} = \langle x,y \rangle / [x,y] = x$, determine $U(\mathfrak{g})$. Guess a generalization.

{Ribbon knots} = {uα : α ∈ K(0-0), dα = 00}



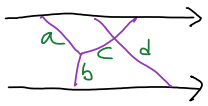
Algebraic knot Theory:



So

$$Z(\{\text{Ribbon knots}\}) \subset \{u\alpha : d\alpha = Z(00)\} \subset A(0\text{-}0)$$

Low algebra. $A(\uparrow\uparrow) \to U(\mathfrak{g})^{\otimes 2}$ via



$$\longrightarrow \sum_{a\text{-}d} f_{abc} \begin{pmatrix} x_a x_d & x_c \\ \otimes & \\ x_b & x_d \end{pmatrix}$$

& likewise, $A(\uparrow_n) \to U(\mathfrak{g})^{\otimes n} \Rightarrow$ $A(\uparrow_n)$ is "universal universal rep. theory"!

VI

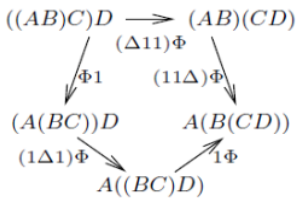
An Associator:

Quantum Algebra's "root object"

$$(AB)C \xrightarrow{\Phi \in U(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 1 1)\Phi \cdot (11\Delta)\Phi$$



The hexagon? Never heard of it.

See Also. B-N & Dancso, arXiv: 1103.1896