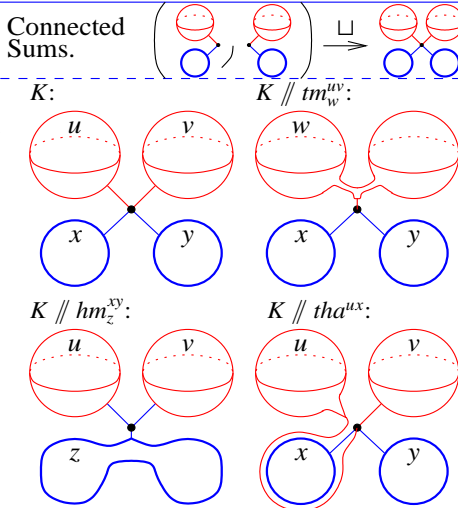
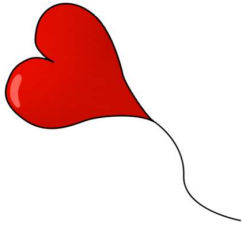


**Operations**

**Punctures & Cuts**

If  $X$  is a space,  $\pi_1(X)$  is a group,  $\pi_2(X)$  is an Abelian group, and  $\pi_1$  acts on  $\pi_2$ .

**Proposition.** The generators generate.



**Definition.**  $l_{xu}$  is the linking number of hoop  $x$  with balloon  $u$ . For  $x \in H$ ,  $\sigma_x := \prod_{u \in T} T_u^{l_{xu}} \in R = R_T = \mathbb{Z}((T_a)_{a \in T})$ , the ring of rational functions in  $T$  variables.

**Theorem 2 [BNS].**  $\exists!$  an invariant  $\beta: w\mathcal{K}^{bh}(H; T) \rightarrow R \times M_{T \times H}(R)$ , intertwining

$$1. \left( \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & A_1 \end{array}, \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & A_2 \end{array} \right) \xrightarrow{\sqcup} \begin{array}{c|cc} \omega_1 \omega_2 & H_1 & H_2 \\ \hline T_1 & A_1 & 0 \\ T_2 & 0 & A_2 \end{array}$$

$$2. \begin{array}{c|c} \omega & H \\ \hline u & \alpha \\ v & \beta \\ T & \Xi \end{array} \xrightarrow{tm_w^{uv}} \begin{array}{c|c} \omega & H \\ \hline w & \alpha + \beta \\ T & \Xi \end{array}_{T_u, T_v \rightarrow T_w}$$

$$3. \begin{array}{c|cc|c} \omega & x & y & H \\ \hline T & \alpha & \beta & \Xi \end{array} \xrightarrow{hm_z^{xy}} \begin{array}{c|c|c} \omega & z & H \\ \hline T & \alpha + \sigma_x \beta & \Xi \end{array}$$

$$4. \begin{array}{c|c|c} \omega & x & H \\ \hline u & \alpha & \theta \\ T & \phi & \Xi \end{array} \xrightarrow{v: 1+\alpha} \begin{array}{c|c|c} v\omega & x & H \\ \hline u & \sigma_x \alpha / v & \sigma_x \theta / v \\ T & \phi / v & \Xi - \phi \theta / v \end{array}$$

and satisfying  $(\epsilon_x; \epsilon_u; \rho_{ux}^\pm) \xrightarrow{\beta} \left( \begin{array}{c|c} 1 & x \\ \hline u & \end{array}; \begin{array}{c|c} 1 & \\ \hline u & T_u^{\pm 1} - 1 \end{array} \right)$ .

**Proposition.** If  $T$  is a u-tangle and  $\beta(\delta T) = (\omega, A)$ , then  $\gamma(T) = (\omega, \sigma - A)$ , where  $\sigma = \text{diag}(\sigma_a)_{a \in S}$ . Under this,  $m_c^{ab} \leftrightarrow tha^{ab} // tm_c^{ab} // hm_c^{ab}$ .

**References.**

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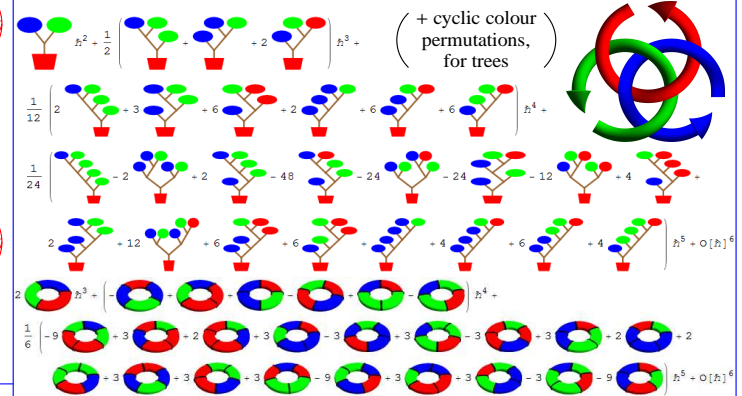
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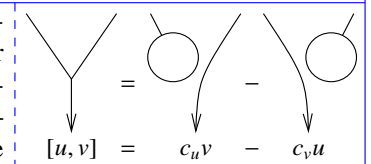
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**Theorem 3 [BND, BN].**  $\exists!$  a homomorphic expansion, aka a homomorphic universal finite type invariant  $Z$  of  $w$ -knotted balloons and hoops.  $\zeta := \log Z$  takes values in  $FL(T)^H \times CW(T)$ .

$\zeta$  is computable!  $\zeta$  of the Borromean tangle, to degree 5:



**Proposition [BN].** Modulo all relations that universally hold for the 2D non-Abelian Lie algebra and after some changes-of-variable,  $\zeta$  reduces to  $\beta$  and the KBH operations on  $\zeta$  reduce to the formulas in Theorem 2.



**A Big Question.** Does it all extend to arbitrary 2-knots (not necessarily “simple”)? To arbitrary codimension-2 knots?

**BF Following [CR].**  $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g})$ ,  $B \in \Omega^2(M, \mathfrak{g}^*)$ ,

$$S(A, B) := \int_M \langle B, F_A \rangle.$$

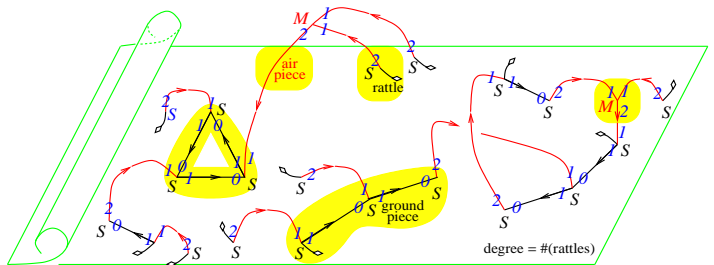
With  $\kappa: (S = \mathbb{R}^2) \rightarrow M$ ,  $\beta \in \Omega^0(S, \mathfrak{g})$ ,  $\alpha \in \Omega^1(S, \mathfrak{g}^*)$ , set

$$O(A, B, \kappa) := \int \mathcal{D}\beta \mathcal{D}\alpha \exp\left(\frac{i}{\hbar} \int_S \langle \beta, d_{\kappa^*} \alpha + \kappa^* B \rangle\right).$$

**The BF Feynman Rules.** For an edge  $e$ , let  $\Phi_e$  be its direction, in  $S^3$  or  $S^1$ . Let  $\omega_3$  and  $\omega_1$  be volume forms on  $S^3$  and  $S^1$ . Then

$$Z_{BF} = \sum_{\text{diagrams } D} \frac{|D|}{|\text{Aut}(D)|} \int_{\mathbb{R}^2} \dots \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \dots \int_{\mathbb{R}^4} \prod_{e \in D} \Phi_e^* \omega_3 \prod_{e \in D} \Phi_e^* \omega_1$$

(modulo some  $STU$ - and  $IHX$ -like relations).



**Issues.** • Signs don't quite work out, and BF seems to reproduce only “half” of the wheels invariant.

• There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.

• I don't know how to define “finite type” for arbitrary 2-knots.



“God created the knots, all else in topology is the work of mortals.”

Leopold Kronecker (modified)

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