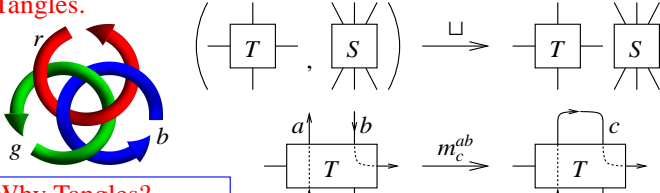


Abstract. I will describe some very good formulas for a (matrix plus scalar)-valued extension of the Alexander polynomial to tangles, then say that everything extends to virtual tangles, then roughly to simply knotted balloons and hoops in 4D, then the target space extends to (free Lie algebras plus cyclic words), and the result is a universal finite type of the knotted objects in its domain. Taking a cue from the BF topological quantum field theory, everything should extend (with some modifications) to arbitrary codimension-2 knots in arbitrary dimension and in particular, to arbitrary 2-knots in 4D. But what is really going on is still a mystery.

Tangles.



Why Tangles?

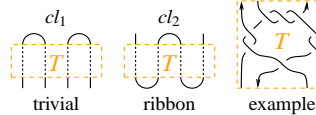
- Finitely presented. (meta-associativity: $m_c^{ab} // m_c^{ac} = m_c^{bc} // m_c^{ab}$)

- Divide and conquer proofs and computations.

- “Algebraic Knot Theory”: If K is ribbon,

$$Z(K) \in \{cl_2(Z): cl_1(Z) = 1\}.$$

(Genus and crossing number are also definable properties).



Theorem 1. $\exists!$ an invariant $\gamma: \{\text{pure framed } S\text{-component tangles}\} \rightarrow R \times M_{S \times S}(R)$, where $R = R_S = \mathbb{Z}((T_a)_{a \in S})$ is the ring of rational functions in S variables, intertwining

$$1. \left(\begin{array}{c|c} \omega_1 & S_1 \\ \hline S_1 & A_1 \end{array}, \begin{array}{c|c} \omega_2 & S_2 \\ \hline S_2 & A_2 \end{array} \right) \xrightarrow{\sqcup} \begin{array}{c|cc} \omega_1 \omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$$

$$2. \begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[\mu := 1 - \beta]{m_c^{ab}} \begin{array}{c|ccc} \mu \omega & c & S & \\ \hline c & \gamma + \alpha \delta / \mu & \epsilon + \delta \theta / \mu \\ S & \phi + \alpha \psi / \mu & \Xi + \psi \theta / \mu \end{array}_{T_a, T_b \rightarrow T_c}$$

and satisfying $(|a; a \nearrow b, b \nearrow a) \xrightarrow{\gamma} \left(\begin{array}{c|c} 1 & a \\ \hline a & 1 \end{array}; \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1 - T_a^{\pm 1} \\ & 0 & T_a^{\pm 1} \end{array} \right)$

In Addition, • This is really “just” a stitching formula for Burau/Gassner [LD, KLW, CT].

- $L \mapsto \omega$ is Alexander, mod units.
- $L \mapsto (\omega, A) \mapsto \omega \det'(A - I) / (1 - T')$ is the MVA, mod units.
- The “fastest” Alexander algorithm.
- There are also formulas for strand deletion, reversal, and doubling.
- Every step along the computation is the invariant of something.
- Extends to and more naturally defined on v/w-tangles.
- Fits in one column, including propaganda & implementation.



Implementation key idea:

```

(ω, A = (αab)) ↔
(ω, λ = ∑ αab ha hb)

F := F[ω1, λ1] F[ω2, λ2] := F[ω1ω2, λ1λ2];
ma,b := F[ω, λ] := Module[α, β, γ, δ, ε, φ, ψ, Ξ, μ],
(α β θ) = (σα1, ha, λ σα2, hb, λ σα3, λ);
(γ δ ε) = (σγ1, ha, λ σγ2, hb, λ σγ3, λ);
(φ ψ Ξ) = (σφ1, ha, λ σφ2, hb, λ σφ3, λ);
Γ[μ = 1 - β] ω, {tc, 1}. (γ + αδ/μ ε + δθ/μ);
(φ + αψ/μ Ξ + ψθ/μ). {hc, 1}

M = Outer[Factor[σα1, ha, λ], S, S];
M = Prepend[M, ta & /@ S] // Transpose;
M = Prepend[M, Prepend[ha & /@ S, ω]];
M // MatrixForm];
    
```

Meta-Associativity

$$\gamma = \Gamma[\omega, \{t_1, t_2, t_3, t_s\} \cdot \left\{ \begin{array}{cccc} \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{array} \right\} \cdot \{h_1, h_2, h_3, h_s\}];$$

$$(\gamma // m_{12 \rightarrow 1} // m_{13 \rightarrow 1}) = (\gamma // m_{23 \rightarrow 2} // m_{12 \rightarrow 1})$$

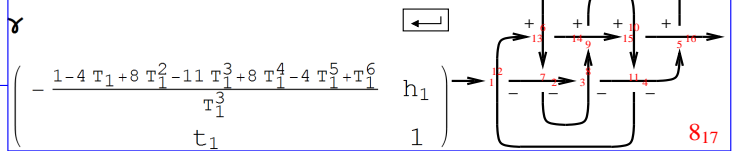
True R3

$$\{Rm_{51} Rm_{62} Rp_{34} // m_{14 \rightarrow 1} // m_{25 \rightarrow 2} // m_{36 \rightarrow 3}, Rp_{61} Rm_{24} Rm_{35} // m_{14 \rightarrow 1} // m_{25 \rightarrow 2} // m_{36 \rightarrow 3}\}$$

$$\left\{ \begin{array}{ccc} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{array} \right\}, \left\{ \begin{array}{ccc} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & \frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{array} \right\}$$

$$\gamma = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15};$$

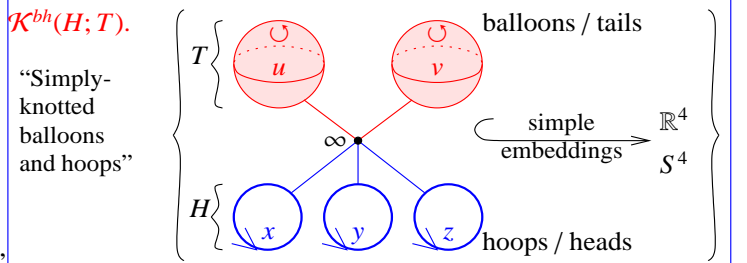
$$Do[\gamma = \gamma // m_{1k \rightarrow 1}, \{k, 2, 16\}];$$



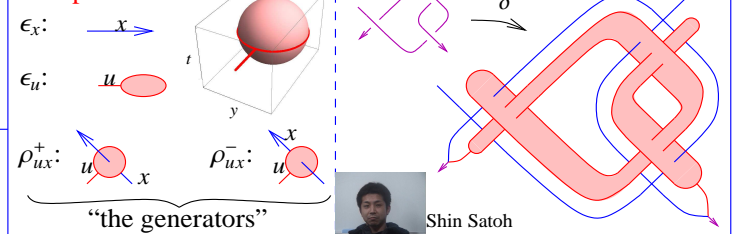
Weaknesses, • m_c^{ab} is non-linear.

- The product ωA is always Laurent, but proving this takes induction with exponentially many conditions.

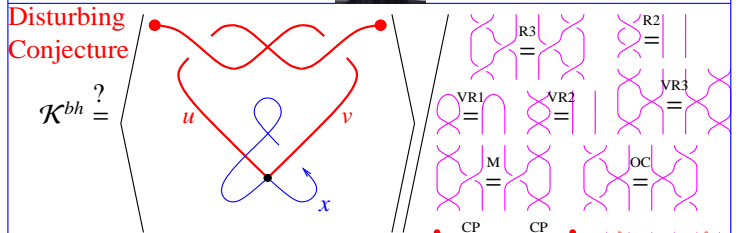
$\mathcal{K}^{bh}(H; T)$



Examples.



Disturbing Conjecture



Dictionary.

