

Simple 2-Knots.

“broken surface diagram”
 A 4D knot by Carter and Saito [CS]

$\omega\epsilon\beta/F$

Dalvit
 $\omega\epsilon\beta/Dal$

Question. Does it all extend to arbitrary 2-knots (not necessarily “simple”)? To arbitrary codimension-2 knots?

BF Following [CR]. $A \in \Omega^1(M = \mathbb{R}^4, \mathfrak{g}), B \in \Omega^2(M, \mathfrak{g}^*),$

$$S(A, B) := \int_M \langle B, F_A \rangle.$$

With $\kappa: (S = \mathbb{R}^2) \rightarrow M, \beta \in \Omega^0(S, \mathfrak{g}), \alpha \in \Omega^1(S, \mathfrak{g}^*),$ set

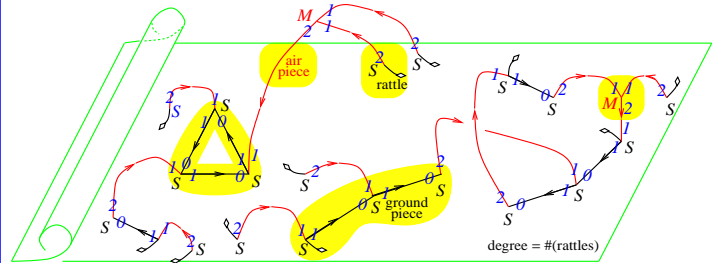
$$O(A, B, \kappa) := \int \mathcal{D}\beta \mathcal{D}\alpha \exp\left(\frac{i}{\hbar} \int_S \langle \beta, d_{\kappa^* A} \alpha + \kappa^* B \rangle\right).$$

The BF Feynman Rules. For an edge $e,$ let Φ_e be its direction, in S^3 or $S^1.$ Let ω_3 and ω_1 be volume forms on S^3 and $S^1.$ Then

$$Z_{BF} = \sum_{\text{diagrams } D} \frac{|D|}{|\text{Aut}(D)|} \int_{\mathbb{R}^2} \dots \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} \dots \int_{\mathbb{R}^4} \prod_{\text{red } e \in D} \Phi_e^* \omega_3 \prod_{\text{black } e \in D} \Phi_e^* \omega_1$$

(modulo some IHX-like relations).

See also [Wa]



Issues. • Signs don't quite work out, and BF seems to reproduce only “half” of the wheels invariant on simple 2-knots.

- There are many more configuration space integrals than BF Feynman diagrams and than just trees and wheels.
- I don't know how to define / analyze “finite type” for general 2-knots.
- I don't know how to reduce Z_{BF} to combinatorics / algebra.

References.

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The Generators

“the crossing” $\omega\epsilon\beta/X$

“v-xing” $\omega\epsilon\beta/vX$

“cap” δ

The Double Inflation Procedure $\delta.$

w-Knots.

$w\mathcal{K} := PA \langle \dots \rangle$

Is this All???

OC: \dots as \dots yet not UC: \dots

A Big Open Problem. δ maps w-knots onto simple 2-knots. To what extent is it a bijection? What other relations are required? In other words, **find a simple description of simple 2-knots.**

The Full 2-Knot Story

Rewrites of IHX.

Riddles, in case you are bored.

- Can you find uncountably many distinct subsets $\{A_\alpha\}$ of \mathbb{Z} such that whenever $\alpha \neq \beta$ either $A_\alpha \subset A_\beta$ or $A_\beta \subset A_\alpha$?
- Can you find uncountably many distinct subsets $\{B_\alpha\}$ of \mathbb{Z} such that whenever $\alpha \neq \beta$ the intersection $B_\alpha \cap B_\beta$ is finite?

Even better,

“God created the knots, all else in topology is the work of mortals.”
 Leopold Kronecker (modified)

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