

Meta-Groups, Meta-Bicrossed-Products, and the Alexander Polynomial, 2

A **Meta-Bicrossed-Product** is a collection of sets $\beta(\eta, \tau)$ and operations tm_w^{uv} , hm_z^{xy} and sw_{ux}^{th} (and lesser ones), such that tm and hm are “associative” and (1) and (2) hold (+ lesser conditions). A meta-bicrossed-product defines a meta-group with $G_\gamma := \beta(\gamma, \gamma)$ and gm as in (3).

Example. Take $\beta(\eta, \tau) = M_{\tau \times \eta}(\mathbb{Z})$ with row operations for the tails, column operations for the heads, and a trivial swap.

β Calculus. Let $\beta(\eta, \tau)$ be

$$\left\{ \begin{array}{c|ccc} \omega & h_1 & h_2 & \dots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle| \begin{array}{l} h_j \in \eta, t_i \in \tau, \text{ and } \omega \text{ and} \\ \text{the } \alpha_{ij} \text{ are rational functions} \\ \text{in a variable } X \end{array} \right\},$$

$$tm_w^{uv} : \begin{array}{c|c} \omega & \dots \\ \hline t_u & \alpha \\ \hline t_v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \dots \\ \hline t_w & \alpha + \beta \\ \vdots & \gamma \end{array}, \quad \begin{array}{c|c} \omega_1 & \eta_1 \\ \hline \tau_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & \eta_2 \\ \hline \tau_2 & \alpha_2 \end{array} = \begin{array}{c|c} \omega_1\omega_2 & \eta_1 \quad \eta_2 \\ \hline \tau_1 & \alpha_1 \quad 0 \\ \tau_2 & 0 \quad \alpha_2 \end{array},$$

$$hm_z^{xy} : \begin{array}{c|ccc} \omega & h_x & h_y & \dots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|ccc} \omega & h_z & \dots & \\ \hline \vdots & \alpha + \beta + \langle \alpha \rangle \beta & \gamma & \end{array},$$

$$sw_{ux}^{th} : \begin{array}{c|ccc} \omega & h_x & \dots & \\ \hline t_u & \alpha & \beta & \vdots \\ \vdots & \gamma & \delta & \vdots \end{array} \mapsto \begin{array}{c|ccc} \omega \epsilon & h_x & \dots & \\ \hline t_u & \alpha(1 + \langle \gamma \rangle / \epsilon) & \beta(1 + \langle \gamma \rangle / \epsilon) & \vdots \\ \vdots & \gamma / \epsilon & \delta - \gamma \beta / \epsilon & \vdots \end{array},$$

where $\epsilon := 1 + \alpha$ and $\langle c \rangle := \sum_i c_i$, and let

$$R_{ab}^p := \begin{array}{c|cc} 1 & h_a & h_b \\ \hline t_a & 0 & X - 1 \\ \hline t_b & 0 & 0 \end{array} \quad R_{ab}^m := \begin{array}{c|cc} 1 & h_a & h_b \\ \hline t_a & 0 & X^{-1} - 1 \\ \hline t_b & 0 & 0 \end{array}.$$

Theorem. Z^β is a tangle invariant (and more). Restricted to knots, the ω part is the Alexander polynomial. On braids, it is equivalent to the Burau representation. A variant for links contains the multivariable Alexander polynomial.

Why Happy? • Applications to w-knots.

- Everything that I know about the Alexander polynomial can be expressed cleanly in this language (even if without proof), except HF, but including genus, ribboness, cabling, v-knots, knotted graphs, etc., and there’s potential for vast generalizations.
- The least wasteful “Alexander for tangles” I’m aware of.

- Every step along the computation is the invariant of something.
- Fits on one sheet, including implementation & propaganda.



Further meta-monoids. Π (and variants), \mathcal{A} (and quotients), vT , ...

Further meta-bicrossed-products. Π (and variants), $\vec{\mathcal{A}}$ (and quotients), M_0 , M , \mathcal{K}^{bh} , \mathcal{K}^{rbh} , ...

Meta-Lie-algebras. \mathcal{A} (and quotients), \mathcal{S} , ...

Meta-Lie-bialgebras. $\vec{\mathcal{A}}$ (and quotients), ...

I don’t understand the relationship between gr and H , as it appears, for example, in braid theory.

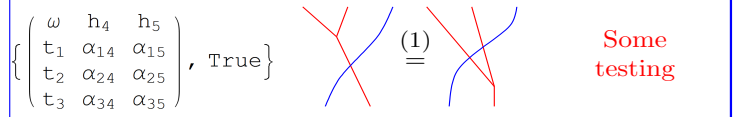
I mean business!

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BSimp = Factor; SetAttributes[BSimp, Listable];
BCollect[B[ω, A_]] := B[BSimp[ω],
Collect[A, h_], Collect[σ, t_], BSimp[ε]];
BForm[B[ω, A_]] := Module[{ts, hs, M},
ts = Union[Cases[B[ω, A], t_ >> u, Infinity]];
hs = Union[Cases[B[ω, A], h_ >> x, Infinity]];
M = Outer[BSimp[Coefficient[A, h_], ts]], hs, ts];
PrependTo[M, ts & /@ ts];
M = Prepend[Transpose[M], Prepend[hs & /@ hs, ω]];
MatrixForm[M];
BForm[else_] := else /. B_B >> BForm[B];
Format[B_B, StandardForm] := BForm[B];
    
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$$\{\beta = B[\omega, \text{Sum}[\alpha_{10+i+j} t_i h_j, \{i, \{1, 2, 3\}\}, \{j, \{4, 5\}\}]\},$$

$$(\beta // tm_{12 \rightarrow 1} // sw_{14}) = (\beta // sw_{24} // sw_{14} // tm_{12 \rightarrow 1})$$



$$\left\{ \begin{array}{c|ccc} \omega & h_4 & h_5 \\ \hline t_1 & \alpha_{14} & \alpha_{15} \\ t_2 & \alpha_{24} & \alpha_{25} \\ t_3 & \alpha_{34} & \alpha_{35} \end{array} \right\}, \text{ True}$$

$$\left\{ \begin{array}{c|ccc} 1 & h_1 & h_2 \\ t_2 & -\frac{1+X}{X} & 0 \\ t_3 & -\frac{1+X}{X} & -\frac{1+X}{X} \end{array} \right\}, \left\{ \begin{array}{c|ccc} 1 & h_1 & h_2 \\ t_2 & -\frac{1+X}{X} & 0 \\ t_3 & -\frac{1+X}{X} & -\frac{1+X}{X} \end{array} \right\}$$

... divide and conquer!

$$\beta = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15}$$

$$\left(\begin{array}{ccccccccccc} 1 & h_1 & h_3 & h_5 & h_7 & h_9 & h_{11} & h_{13} & h_{15} \\ t_2 & 0 & 0 & 0 & -\frac{1+X}{X} & 0 & 0 & 0 & 0 \\ t_4 & 0 & 0 & 0 & 0 & 0 & -\frac{1+X}{X} & 0 & 0 \\ t_6 & 0 & 0 & 0 & 0 & 0 & 0 & -1+X & 0 \\ t_8 & 0 & -\frac{1+X}{X} & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+X \\ t_{12} & -\frac{1+X}{X} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{14} & 0 & 0 & 0 & 0 & -1+X & 0 & 0 & 0 \\ t_{16} & 0 & 0 & -1+X & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Do[β = β // gm_{1k-1}, {k, 2, 10}]; β

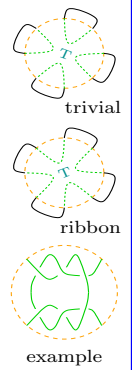
$$\left(\begin{array}{cccc} \frac{1}{X} & h_1 & h_{11} & h_{13} & h_{15} \\ t_1 & -\frac{(-1+X)(1+X)}{X} & -(-1+X)(1-X+X^2) & (-1+X)(1-X+X^2) & -1+X \\ t_{12} & -\frac{1+X}{X} & 0 & 0 & 0 \\ t_{14} & -1+X & \frac{(-1+X)^2(1-X+X^2)}{X} & -\frac{(-1+X)^2(1-X+X^2)}{X} & 0 \\ t_{16} & -\frac{1+X}{X} & (-1+X)^2 & -\frac{(-1+X)^3}{X} & 0 \end{array} \right)$$

James Waddell Alexander

$$\text{Do}[\beta = \beta // gm_{1k-1}, \{k, 11, 16\}]; \beta$$

$$\left(-\frac{1-4X+8X^2-11X^3+8X^4-4X^5+X^6}{X^3} \right)$$

- A Partial To Do List.**
1. Where does it more simply come from?
 2. Remove all the denominators.
 3. How do determinants arise in this context?
 4. Understand links (“meta-conjugacy classes”).
 5. Find the “reality condition”.
 6. Do some “Algebraic Knot Theory”.
 7. Categorify.
 8. Do the same in other natural quotients of the v/w-story.



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas
James C. Lagarias