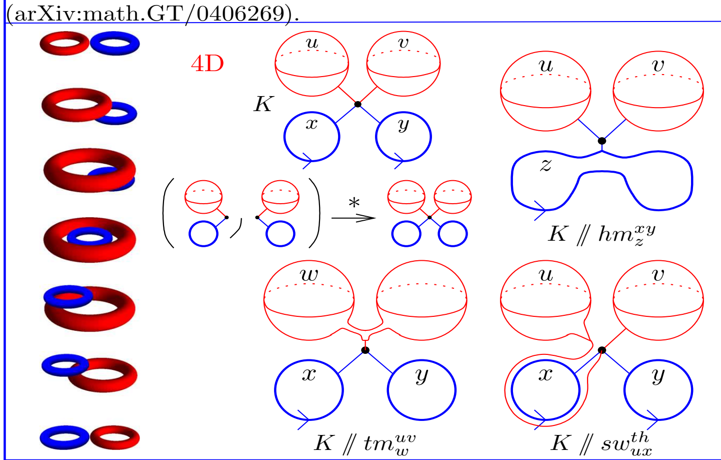
Meta–Groups, Meta–Bicrossed–Products, and the Alexander Polynomial, 1

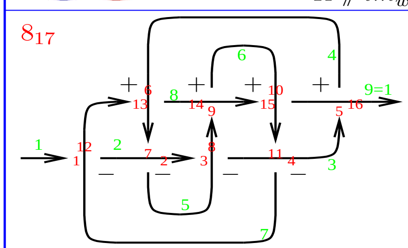
Dror Bar–Natan at Sheffield, February 2013.

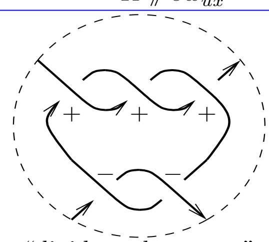
http://www.math.toronto.edu/~drorbn/Talks/Sheffield-130206/

Abstract. I will define "meta-groups" and explain how one specific meta-group, which in itself is a "meta-bicrossed-product", gives rise to an "ultimate Alexander invariant" of tangles, that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that's a wonderful playground.

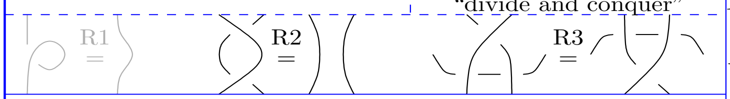
This work is closely related to work by Le Dimet (Comment. Math. Helv. **67** (1992) 306–315), Kirk, Livingston and Wang (arXiv:math/9806035) and Cimasoni and Turaev (arXiv:math.GT/0406269).



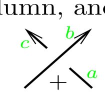




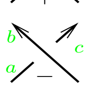
"divide and conquer"



A Standard Alexander Formula. Label the arcs 1 through $(n+1)=1$, make an $n\times n$ matrix as below, delete one row and one column, and compute the determinant:



	a	b	c
c	-1	$1-X$	X



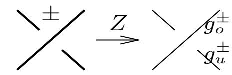
	a	b	c
c	$-X$	$X-1$	1

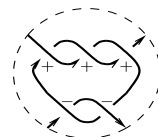
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & X-1 & 0 & -X \\ -1 & X & 0 & 0 & 0 & 0 & 1-X & 0 \\ 0 & -1 & X & 0 & 1-X & 0 & 0 & 0 \\ X-1 & 0 & -X & 1 & 0 & 0 & 0 & 0 \\ 0 & 1-X & 0 & -1 & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -X & 1 & 0 & X-1 \\ 0 & 0 & 1-X & 0 & 0 & -1 & X & 0 \\ 0 & 0 & 0 & X-1 & 0 & 0 & -X & 1 \end{pmatrix} \texttt{[[1 ;; 7, 1 ;; 7]] // Det}$$

$-1 + 4X - 8X^2 + 11X^3 - 8X^4 + 4X^5 - X^6$

Alexander Issues.
- Quick to compute, but computation departs from topology.
- Extends to tangles, but at an exponential cost.
- Hard to categorify.

Idea. Given a group G and two "YB" pairs $R^\pm = (g_o^\pm, g_u^\pm) \in G^2$, map them to xings and "multiply along", so that

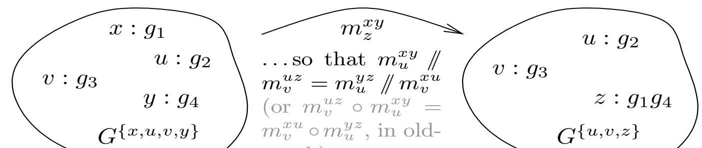




$$\overset{Z}{\longmapsto} \begin{pmatrix} g_o^+ g_u^+ g_o^+ g_u^- g_o^- g_u^+ g_o^+ g_u^+ \\ g_u^- g_o^- \end{pmatrix}$$

This Fails! R2 implies that $g_o^\pm g_o^\mp = e = g_u^\pm g_u^\mp$ and then R3 implies that g_o^+ and g_u^+ commute, so the result is a simple counting invariant.

A Group Computer. Given G, can store group elements and perform operations on them:



…so that $m^{xy}_u \,/\!\!/\, m^{uz}_v = m^{yz}_u \,/\!\!/\, m^{xu}_v$ (or $m^{uz}_v \circ m^{xy}_u = m^{xu}_v \circ m^{yz}_u$, in old-speak).

Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ^z_{xy} for element cloning, ρ^x_y for renamings, and $(D_1, D_2) \mapsto D_1 \cup D_2$ for merging, and many obvious composition axioms relating those. $P = \{x: g_1, y: g_2\} \Rightarrow P = \{d_yP\} \cup \{d_xP\}$

A Meta-Group. Is a similar "computer", only its internal structure is unknown to us. Namely it is a collection of sets $\{G_\gamma\}$ indexed by all finite sets γ, and a collection of operations m^{xy}_z, S_x, e_x, d_x, Δ^z_{xy} (sometimes), ρ^x_y, and $\cup$, satisfying the exact same *linear* properties.

Example 0. The non-meta example, $G_\gamma := G^\gamma$.

Example 1. $G_\gamma := M_{\gamma\times\gamma}(\mathbb{Z})$, with simultaneous row and column operations, and "block diagonal" merges. Here if $P = \begin{pmatrix} x: & a & b \\ y: & c & d \end{pmatrix}$ then $d_yP = (x: a)$ and $d_xP = (y: d)$ so $\{d_yP\} \cup \{d_xP\} = \begin{pmatrix} x: & a & 0 \\ y: & 0 & d \end{pmatrix} \neq P$. So this G is truly meta.

Claim. From a meta-group G and YB elements $R^\pm \in G_2$ we can construct a knot/tangle invariant.

Bicrossed Products. If $G = HT$ is a group presented as a product of two of its subgroups, with $H \cap T = \{e\}$, then also $G = TH$ and G is determined by H, T, and the "swap" map $sw^{th}: (t,h) \mapsto (h', t')$ defined by $th = h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw: T\times H \to H\times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H \times T$, the "bicrossed product".