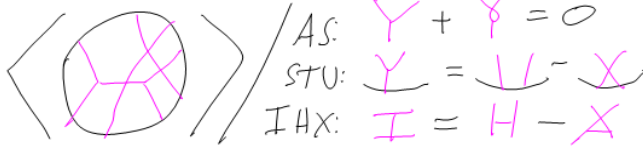


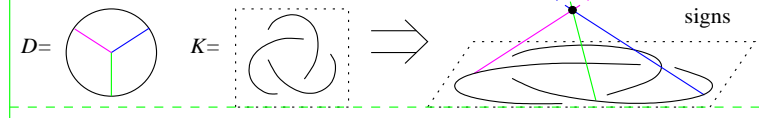
Day 3: Chern-Simons, Gaussian Integration, Feynman Diagrams

Cosmic Coincidences

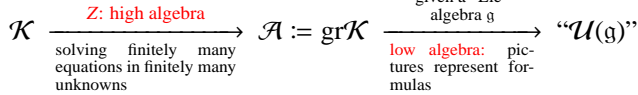
Recall. $\mathcal{K} = \{\text{knots}\}$, $\mathcal{A} := \text{gr}\mathcal{A} = \mathcal{D}/\text{rels} =$



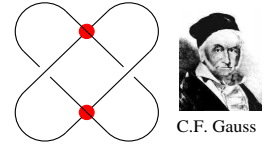
$\langle D, K \rangle_{\mathbb{N}} :=$ (The signed Stonehenge) :
pairing of D and K



Seek $Z: \mathcal{K} \rightarrow \hat{\mathcal{A}}$ such that if K is n -singular, $Z(K) = D_k + \dots$

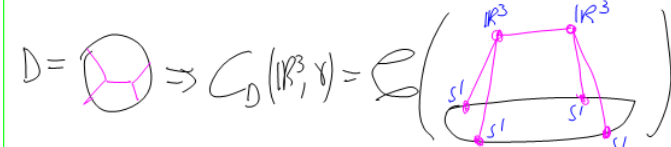


The Gaussian linking number = $\langle \text{vertical chopsticks}, \text{knot} \rangle_{\mathbb{N}}$



The generating function of all cosmic coincidences:

$$Z(K) := \lim_{N \rightarrow \infty} \sum_{\text{3-valent } D} \frac{\langle D, K \rangle_{\mathbb{N}} D}{2^c c! \binom{N}{c}} \in \mathcal{A}$$



Theorem. Given a parametrized knot γ in \mathbb{R}^3 , up to renormalizing the “framing anomaly”,

$$Z(\gamma) = \sum_{D \in \mathcal{D}} \frac{C(D)D}{|\text{Aut}(D)|} \int_{C_D(\mathbb{R}^3, \gamma)} \bigwedge_{e \in E(D)} \phi_e^* \omega \in \mathcal{A}$$

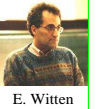
is an expansion. Here \mathcal{D} is the set of all “Feynman diagrams”, $E(D)$ is the set of internal edges (and chords) of D , $C_D(\mathbb{R}^3, \gamma)$ is the configuration space of placements of D on/around γ , $\phi: C_D(\mathbb{R}^3, \gamma) \rightarrow (S^2)^{E(D)}$ is the “direction of the edges” map, and ω is a volume form on S^2 .

Claim. It all comes from the Chern-Simons-Witten theory,

$$\int_{A \in \Omega^1(\mathbb{R}^3, \mathfrak{g})} \text{tr}_R \text{hol}_\gamma(A) \exp \left[\frac{ik}{4\pi} \int_{\mathbb{R}^3} \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right],$$

where $\Omega^1(\mathbb{R}^3, \mathfrak{g})$ is the space of all \mathfrak{g} -valued 1-forms on \mathbb{R}^3 (really, connections), k is some large constant, R is some representation of \mathfrak{g} and tr_R is trace in R , and $\text{hol}_\gamma(A)$ is the holonomy of A along γ .

References. Witten’s *Quantum field theory and the Jones polynomial*, Axelrod-Singer’s *Chern-Simons perturbation theory I-II*, D. Thurston’s [arXiv:math.QA/9901110](https://arxiv.org/abs/math/9901110), Polyak’s [arXiv:math.GT/0406251](https://arxiv.org/abs/math.GT/0406251), and my videotaped 2014 class ω/AKT .

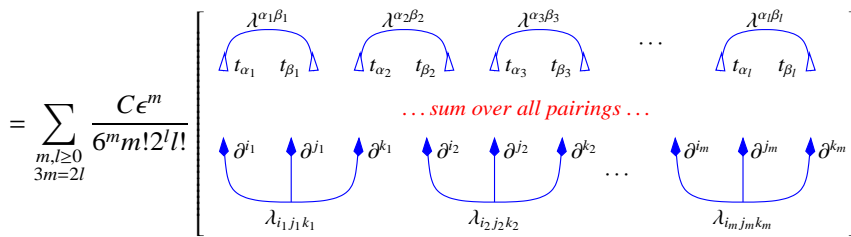


Gaussian Integration. (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of \mathbb{R}^n , let $(t_i)_{i=1}^n$ be a set of “dual” variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2} \lambda_{ij} x^i x^j + \frac{c}{6} \lambda_{ijk} x^i x^j x^k} = \sum_{m \geq 0} \frac{\epsilon^m}{6^m m!} \int_{\mathbb{R}^n} (\lambda_{ijk} x^i x^j x^k)^m e^{-\frac{1}{2} \lambda_{ij} x^i x^j}$$

$$= \sum_{m \geq 0} \frac{C \epsilon^m}{6^m m!} (\lambda_{ijk} \partial^i \partial^j \partial^k)^m e^{\frac{1}{2} \lambda^{\alpha\beta} t_\alpha t_\beta} \Big|_{t_\alpha=0} = \sum_{\substack{m, l \geq 0 \\ 3m=2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} (\lambda_{ijk} \partial^i \partial^j \partial^k)^m (\lambda^{\alpha\beta} t_\alpha t_\beta)^l$$

Feynman



$$= \sum_{\substack{m, l \geq 0 \\ 3m=2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} \sum_{\substack{m\text{-vertex fully marked} \\ \text{Feynman diagrams } D}} \mathcal{E}(D)$$

$$\lambda_{i_1 j_1 k_1} \lambda_{i_2 j_2 k_2} \lambda^{i_1 i_2} \lambda^{j_1 j_2} \lambda^{k_1 k_2} \quad \lambda_{i_1 j_1 k_1} \lambda_{i_2 j_2 k_2} \lambda^{i_1 j_1} \lambda^{k_1 k_2} \lambda^{i_2 j_2} \text{ etc.}$$

$$= C \sum_{\text{unmarked Feynman diagrams } D} \frac{\epsilon^{m(D)} \mathcal{E}(D)}{|\text{Aut}(D)|}$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m! 2^l l!}{|\text{Aut}(D)|}$.

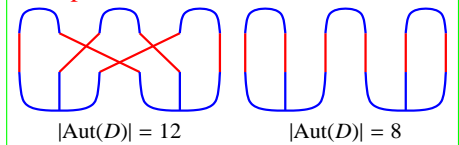
Proof of the Claim. The group $G_{m,l} := [(S_3)^m \times S_m] \times [(S_2)^l \times S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D , and the stabilizer of any given P is $\text{Aut}(D)$. \square

The Fourier Transform.

$(F: V \rightarrow \mathbb{C}) \Rightarrow (\tilde{f}: V^* \rightarrow \mathbb{C})$ via $\tilde{F}(\varphi) := \int_V f(v) e^{-i\langle \varphi, v \rangle} dv$. Some facts:

- $\tilde{f}(0) = \int_V f(v) dv$.
- $\frac{\partial}{\partial \varphi_i} \tilde{f} \sim \tilde{v}^i f$.
- $(e^{Q/2}) \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v \rangle$ for $L: V \rightarrow V^*$, and $Q^{-1}(\varphi) := \langle \varphi, L^{-1}\varphi \rangle$. (This is the key point in the proof of the Fourier inversion formula!)

Examples.



Monsters left to Slay.

- Convergence.
- Proof of invariance.
- The framing anomaly.
- Universality.
- d^{-1} doesn’t really exist, Faddeev-Popov, determinants, ghosts, Berezin integration.
- Assembly.

