

# Trees and Wheels and Balloons and Hoops

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$\omega\epsilon\beta := \text{http://www.math.toronto.edu/~drorbn/Talks/Zurich-130919}$



## 15 Minutes on Algebra

Let  $T$  be a finite set of “tail labels” and  $H$  a finite set of “head labels”. Set

$$M_{1/2}(T; H) := FL(T)^H,$$

“ $H$ -labeled lists of elements of the degree-completed free Lie algebra generated by  $T$ ”.

$$FL(T) = \left\{ 2t_2 - \frac{1}{2}[t_1, [t_1, t_2]] + \dots \right\} / \left( \begin{array}{c} \text{anti-symmetry} \\ \text{Jacobi} \end{array} \right) \dots \text{with the obvious bracket.}$$

$$M_{1/2}(u, v; x, y) = \left\{ \lambda = \left( x \rightarrow \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ x \end{array}, y \rightarrow \begin{array}{c} v \quad u \\ \diagdown \quad \diagup \\ y \end{array} - \frac{22}{7} \begin{array}{c} u \quad v \\ \diagdown \quad \diagup \\ y \end{array} \right) \dots \right\}$$

## Operations $M_{1/2} \rightarrow M_{1/2}$ .

**Tail Multiply**  $tm_w^{uv}$  is  $\lambda \mapsto \lambda \parallel (u, v \rightarrow w)$ , satisfies “meta-associativity”,  $tm_u^{uv} \parallel tm_u^{uv} = tm_v^{uv} \parallel tm_u^{uv}$ .

**Head Multiply**  $hm_z^{xy}$  is  $\lambda \mapsto (\lambda \setminus \{x, y\}) \cup (z \rightarrow \text{bch}(\lambda_x, \lambda_y))$ , where

$$\text{bch}(\alpha, \beta) := \log(e^\alpha e^\beta) = \alpha + \beta + \frac{[\alpha, \beta]}{2} + \frac{[\alpha, [\alpha, \beta]] + [[\alpha, \beta], \beta]}{12} + \dots$$

satisfies  $\text{bch}(\text{bch}(\alpha, \beta), \gamma) = \log(e^{\alpha} e^{\beta} e^{\gamma}) = \text{bch}(\alpha, \text{bch}(\beta, \gamma))$  and hence meta-associativity,  $hm_x^{xy} \parallel hm_x^{xz} = hm_y^{yz} \parallel hm_x^{xy}$ .

**Tail by Head Action**  $tha^{ux}$  is  $\lambda \mapsto \lambda \parallel RC_u^{\lambda_x}$ , where  $C_u^{-\gamma}: FL \rightarrow FL$  is the substitution  $u \rightarrow e^{-\gamma} u e^{\gamma}$ , or more precisely,

$$C_u^{-\gamma}: u \rightarrow e^{-\text{ad} \gamma}(u) = u - [\gamma, u] + \frac{1}{2}[\gamma, [\gamma, u]] - \dots,$$

and  $RC_u^\gamma = (C_u^{-\gamma})^{-1}$ . Then  $C_u^{\text{bch}(\alpha, \beta)} = C_u^\alpha \parallel RC_u^{-\beta} \parallel C_u^\beta$  hence  $RC_u^{\text{bch}(\alpha, \beta)} = RC_u^\alpha \parallel RC_u^\beta \parallel RC_u^\alpha$  hence “meta  $u^{xy} = (u^x)^y$ ”,

$$hm_z^{xy} \parallel tha^{uz} = tha^{ux} \parallel tha^{uy} \parallel hm_z^{xy},$$

and  $tm_w^{uv} \parallel C_w^\gamma \parallel tm_w^{uv} = C_w^\gamma \parallel RC_w^{-\gamma} \parallel C_w^\gamma \parallel tm_w^{uv}$  and hence “meta  $(uv)^x = u^x v^x$ ”,  $tm_w^{uv} \parallel tha^{wx} = tha^{ux} \parallel tha^{vx} \parallel tm_w^{uv}$ .

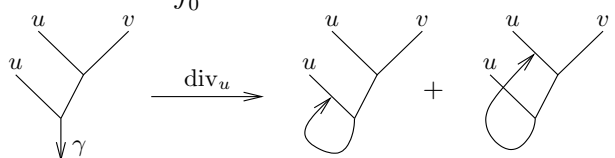
**Wheels.** Let  $M(T; H) := M_{1/2}(T; H) \times CW(T)$ , where  $CW(T)$  is the (completed graded) vector space of cyclic words on  $T$ , or equally well, on  $FL(T)$ :



**Operations.** On  $M(T; H)$ , define  $tm_w^{uv}$  and  $hm_z^{xy}$  as before, and  $tha^{ux}$  by adding some  $J$ -spice:

$$(\lambda; \omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) \parallel RC_u^{\lambda_x},$$

where  $J_u(\gamma) := \int_0^1 ds \text{div}_u(\gamma \parallel RC_u^{s\gamma}) \parallel C_u^{-s\gamma}$ , and



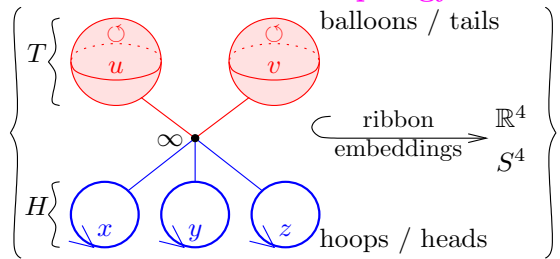
**Theorem Blue.** All blue identities still hold.

**Merge Operation.**  $(\lambda_1; \omega_1) * (\lambda_2; \omega_2) := (\lambda_1 \cup \lambda_2; \omega_1 + \omega_2)$ .

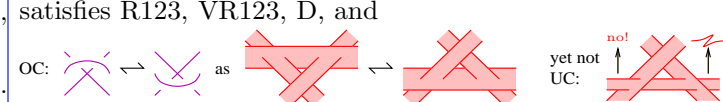
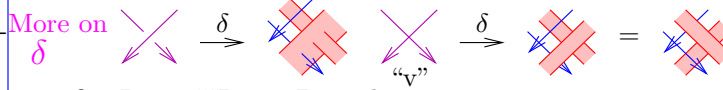
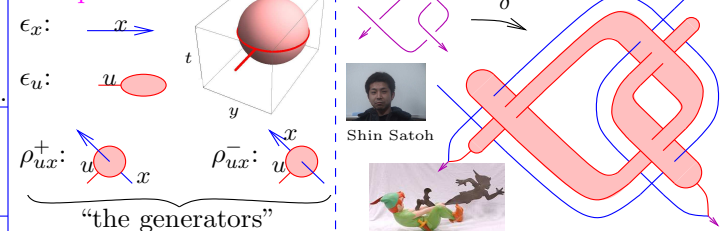
## $\mathcal{K}^{bh}(T; H)$ .

“Ribbon-knotted balloons and hoops”

## 15 Minutes on Topology



## Examples.



- $\delta$  injects u-knots into  $\mathcal{K}^{bh}$  (likely u-tangles too).
- $\delta$  maps v-tangles to  $\mathcal{K}^{bh}$ ; the kernel contains the above and conjecturally (Satoh), that's all.
- Allowing punctures and cuts,  $\delta$  is onto.

## Operations

**Punctures & Cuts**  $\left( \begin{array}{c} \text{balloon} \\ \text{hoop} \end{array} \right) \rightarrow \left( \begin{array}{c} \text{punctured balloon} \\ \text{cut hoop} \end{array} \right)$

If  $X$  is a space,  $\pi_1(X)$  is a group,  $\pi_2(X)$  is an Abelian group, and  $\pi_1$  acts on  $\pi_2$ .

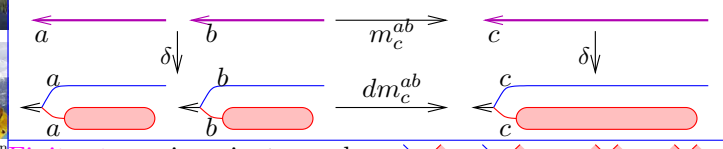
**Riddle.** People often study  $\pi_1(X) = [S^1, X]$  and  $\pi_2(X) = [S^2, X]$ . Why not  $\pi_T(X) := [T, X]$ ?



## “Meta-Group-Action”

- Associativities:  $m_a^{ab} \parallel m_a^{ac} = m_b^{bc} \parallel m_a^{ab}$ , for  $m = tm, hm$ .
- “(uv)<sup>x</sup> = u<sup>x</sup>v<sup>x</sup>”:  $tm_w^{uv} \parallel tha^{wx} = tha^{ux} \parallel tha^{vx} \parallel tm_w^{uv}$ .
- “(u(xy) = (u<sup>x</sup>)y<sup>n</sup>”:  $hm_z^{xy} \parallel tha^{uz} = tha^{ux} \parallel tha^{vy} \parallel hm_z^{xy}$ .

**Tangle concatenations**  $\rightarrow \pi_1 \times \pi_2$ . With  $dm_c^{ab} := tha^{ab} \parallel tm_c^{ab} \parallel hm_c^{ab}$ ,



**Finite type invariants** make sense in the usual way, and “algebra” is (the primitive part of) “gr” of “topology”.