From V to F to KV following [AT].

```
logF = Λ[V]⟦1⟧ // dσ[{x, y} → {y, x}];
logF // EulerE // adSeries[(e^ad - 1)/ad, logF, tb]
```

$\langle \overline{x} \to LS\left[\frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24}\overline{xyy}, -\frac{1}{180}\overline{xxxy} + \frac{1}{80}\overline{xxyy} + \frac{1}{360}\overline{xyyy}, -\frac{1}{720}\overline{xxxyy} + \frac{1}{240}\overline{xxyyy} + \frac{1}{240}\overline{xyxyy} + \frac{1}{720}\overline{xxyxy} - \frac{\overline{xyyyy}}{1440}, \frac{\overline{xxxxxy}}{5040} - \frac{\overline{xxxxyy}}{1344} + \frac{13\,\overline{xxxyyy}}{15120} + \frac{1}{840}\overline{xxyxyy} + \frac{\overline{xxxyxy}}{3360} + \frac{\overline{xxyyyy}}{6720} + \frac{\overline{xyxyyy}}{1260} + \frac{\overline{xxyyxy}}{1680} - \frac{\overline{xyyyyy}}{10080}, \ldots\right],$

$\overline{y} \to LS\left[0, \frac{\overline{xy}}{12}, \frac{1}{24}\overline{xyy}, -\frac{1}{360}\overline{xxxy} + \frac{1}{120}\overline{xxyy} + \frac{1}{180}\overline{xyyy}, -\frac{1}{720}\overline{xxxyy} + \frac{1}{240}\overline{xxyyy} + \frac{1}{240}\overline{xyxyy} + \frac{1}{720}\overline{xxyxy} - \frac{\overline{xyyyy}}{1440}, \frac{\overline{xxxxxy}}{10080} - \frac{\overline{xxxxyy}}{2016} + \frac{\overline{xxxyyy}}{1890} + \frac{\overline{xxyxyy}}{1120} + \frac{\overline{xxxyxy}}{5040} + \frac{\overline{xxyyyy}}{2520} + \frac{1}{840}\overline{xyxyyy} + \frac{\overline{xxyyxy}}{1260} - \frac{\overline{xyyyyy}}{5040}, \ldots\right]\rangle$

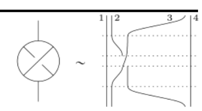
Solving for an associator Φ.

```
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0;
Φs[3, 1, 2] = 1/24; Φ = DKS[3, Φs];
SeriesSolve[Φ,
  (Φ^σ[3,2,1] ≡ -Φ) ∧
   (Φ ** Φ^σ[1,23,4] ** Φ^σ[2,3,4] ≡ Φ^σ[12,3,4] ** Φ^σ[1,2,34])];
Φ (* Can raise degree to 10 *)
```

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 1, 2] → 0}.

$DKS\left[0, \frac{1}{24}\overline{t_{13}t_{23}}, 0, -\frac{7\,\overline{t_{13}t_{23}t_{23}t_{23}}}{5760} + \frac{7\,\overline{t_{13}t_{13}t_{23}t_{23}}}{5760} - \frac{\overline{t_{13}t_{13}t_{13}t_{23}}}{1440}, 0, \frac{31\,\overline{t_{13}t_{23}t_{23}t_{23}t_{23}t_{23}}}{967680} - \frac{157\,\overline{t_{13}t_{13}t_{23}t_{23}t_{13}t_{23}}}{1935360} - \frac{31\,\overline{t_{13}t_{23}t_{13}t_{23}t_{23}t_{23}}}{387072} - \frac{31\,\overline{t_{13}t_{13}t_{23}t_{23}t_{23}t_{23}}}{483840} + \frac{11\,\overline{t_{13}t_{13}t_{13}t_{23}t_{13}t_{23}}}{290304} + \frac{31\,\overline{t_{13}t_{13}t_{23}t_{13}t_{23}t_{23}}}{725760} + \frac{83\,\overline{t_{13}t_{13}t_{13}t_{23}t_{23}t_{23}}}{967680} - \frac{13\,\overline{t_{13}t_{13}t_{13}t_{13}t_{23}t_{23}}}{241920} + \frac{\overline{t_{13}t_{13}t_{13}t_{13}t_{13}t_{23}}}{60480}, \ldots\right]$

The "buckle" Z_B, from Φ.




```
R = DKS[t[1, 2]/2];
Z_B = (-Φ)^σ[13,2,4] ** Φ^σ[1,3,2] ** R^σ[2,3] ** (-Φ)^σ[1,2,3] **
   Φ^σ[12,3,4];
Z_B@{4}
```

$DKS\left[\frac{\overline{t_{23}}}{2}, -\frac{1}{12}\overline{t_{13}t_{23}} - \frac{1}{24}\overline{t_{14}t_{24}} + \frac{1}{24}\overline{t_{14}t_{34}} + \frac{1}{12}\overline{t_{24}t_{34}}, 0, \frac{\overline{t_{13}t_{23}t_{23}t_{23}}}{5760} + \frac{7\,\overline{t_{14}t_{24}t_{24}t_{24}}}{5760} + \frac{\overline{t_{14}t_{34}t_{24}t_{24}}}{1920} - \frac{\overline{t_{14}t_{34}t_{34}t_{24}}}{1920} - \frac{7\,\overline{t_{14}t_{34}t_{34}t_{34}}}{5760} - \frac{\overline{t_{24}t_{34}t_{34}t_{34}}}{5760} + \frac{\overline{t_{14}t_{24}t_{34}t_{24}}}{1920} + \frac{\overline{t_{14}t_{24}t_{14}t_{34}}}{1920} - \frac{\overline{t_{14}t_{34}t_{24}t_{34}}}{1920} - \frac{1}{720}\overline{t_{13}t_{13}t_{23}t_{23}} + \frac{1}{720}\overline{t_{13}t_{13}t_{13}t_{23}} - \frac{7\,\overline{t_{14}t_{14}t_{24}t_{24}}}{5760} + \frac{7\,\overline{t_{14}t_{14}t_{34}t_{34}}}{5760} - \frac{\overline{t_{14}t_{24}t_{34}t_{34}}}{5760} + \frac{\overline{t_{14}t_{14}t_{14}t_{24}}}{1440} - \frac{\overline{t_{14}t_{14}t_{14}t_{34}}}{1440} - \frac{1}{960}\overline{t_{14}t_{14}t_{24}t_{34}} + \frac{\overline{t_{14}t_{24}t_{24}t_{34}}}{5760} - \frac{1}{960}\overline{t_{24}t_{24}t_{34}t_{34}} - \frac{\overline{t_{24}t_{24}t_{24}t_{34}}}{5760}, \ldots\right]$

V from Z_B, following [AET, BND].

```
(El[Z_B // αMap[1, 2, 3, 4], CWS[0]] // Γ // tη^1 // tη^3 //
    hη^2 // hη^4 // hσ[{3} → {2}] // tσ[{2, 4} → {1, 2}])⟦
  1⟧
```

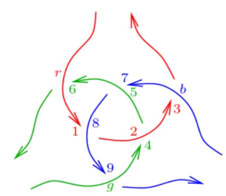
$\langle 1 \to LS\left[0, -\frac{\overline{12}}{24}, 0, \frac{\overline{7\,1\,1\,12}}{5760} - \frac{\overline{7\,1\,12\,2}}{5760} + \frac{\overline{12\,2\,2}}{1440}, 0, -\frac{31\,\overline{1\,1\,1\,1\,12}}{967680} + \frac{31\,\overline{1\,1\,1\,12\,2}}{483840} - \frac{83\,\overline{1\,1\,12\,2\,2}}{967680} - \frac{31\,\overline{1\,12\,12\,2}}{725760} - \frac{31\,\overline{1\,1\,12\,12}}{645120} + \frac{13\,\overline{1\,12\,2\,2\,2}}{241920} + \frac{101\,\overline{12\,12\,2\,2}}{1451520} + \frac{527\,\overline{1\,12\,2\,12}}{5806080} - \frac{\overline{12\,2\,2\,2\,2}}{60480}, \ldots\right],$

$2 \to LS\left[\frac{\overline{1}}{2}, -\frac{\overline{12}}{12}, 0, \frac{\overline{1\,1\,12}}{5760} - \frac{1}{720}\overline{1\,12\,2} + \frac{1}{720}\overline{12\,2\,2}, -\frac{\overline{1\,1\,1\,12}}{7680} + \frac{\overline{1\,1\,12\,2}}{3840} - \frac{\overline{1\,12\,12}}{6912}, -\frac{\overline{1\,1\,1\,1\,12}}{645120} + \frac{23\,\overline{1\,1\,1\,12\,2}}{483840} - \frac{13\,\overline{1\,1\,12\,2\,2}}{161280} - \frac{\overline{1\,12\,12\,2}}{22680} - \frac{41\,\overline{1\,1\,12\,12}}{580608} + \frac{\overline{1\,12\,2\,2\,2}}{15120} + \frac{\overline{12\,12\,2\,2}}{12096} + \frac{71\,\overline{1\,12\,2\,12}}{483840} - \frac{\overline{12\,2\,2\,2\,2}}{30240}, \ldots\right]\rangle$

The Borromean tangle.

```
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
ξ = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
```

```
Do[ξ = ξ // dm[r, k, r], {k, 1, 3}];
Do[ξ = ξ // dm[g, k, g], {k, 4, 6}];
Do[ξ = ξ // dm[b, k, b], {k, 7, 9}];
{ξ⟦1⟧_r@{5}, ξ⟦2⟧@{5}} // Print
```



;

$\{LS\left[0, \overline{bg}, \frac{1}{2}\overline{bbg} + \overline{bgr} + \frac{1}{2}\overline{bgg}, \frac{1}{6}\overline{bbbg} + \frac{1}{2}\overline{bbgr} + \frac{1}{2}\overline{bggr} + \frac{1}{4}\overline{bbgg} + \frac{1}{2}\overline{bgrr} + \frac{1}{6}\overline{bggg}, \frac{1}{24}\overline{bbbbg} + \frac{1}{6}\overline{bbbgr} + \frac{1}{4}\overline{bbggr} + \frac{1}{12}\overline{bbbgg} + \frac{1}{4}\overline{bbgrr} + \frac{1}{6}\overline{bgggr} + \frac{1}{4}\overline{bggrr} - \overline{bbgrg} + \frac{1}{12}\overline{bbggg} - 2\,\overline{bbrgg} + \frac{1}{6}\overline{bgrrr} + \frac{1}{2}\overline{bgbgr} - \overline{bgbrg} - \frac{1}{12}\overline{bbgbg} - \frac{1}{2}\overline{bgrgr} + \frac{1}{24}\overline{bgggg}, \ldots\right],$

$CWS\left[0, 0, 2\,\overline{bgr}, \overline{bbgr} - \overline{bgbr} + \overline{bggr} - \overline{bgrg} + \overline{bgrr} - \overline{brgr}, \frac{\overline{bbbgr}}{3} - \frac{\overline{bbgbr}}{2} + \frac{\overline{bbggr}}{2} + \frac{\overline{bbgrg}}{2} + \frac{\overline{bbgrr}}{2} + \frac{\overline{bbrbg}}{2} - \frac{3\,\overline{bbrgr}}{2} + \frac{\overline{bgbrr}}{2} - \frac{3\,\overline{bggbr}}{2} + \frac{\overline{bgggr}}{3} - \frac{\overline{bggrg}}{2} + \frac{\overline{bggrr}}{2} + \frac{\overline{bgrgg}}{2} - \frac{3\,\overline{bgrrg}}{2} + \frac{\overline{bgrrr}}{3} + \frac{\overline{brggr}}{2} - \frac{\overline{brgrr}}{2} + \frac{\overline{brrgr}}{2}, \ldots\right]\}$

References.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* Annals of Mathematics **175** (2012) 415–463, arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations,* Publications Mathématiques de L'IHÉS, **112-1** (2010) 143–189, arXiv:0903.4067

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-IV,* ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO3, ωεβ/WKO4, and arXiv:1405.1956, arXiv:1405.1955, arXiv: not.yet×2.

Warning. Fidgety!

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/