

1-Smidgen sl_2 Let \mathfrak{g}_1 be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle h, e', l, f \rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with h central and with $[f, l] = f$, $[e', l] = -e'$, and $[e', f] = h - 2\epsilon l$. Over \mathbb{Q} , \mathfrak{g}_1 is a **solvable approximation of sl_2** : $\mathfrak{g}_1 \supset \langle h, e', f, \epsilon h, \epsilon e', \epsilon l, \epsilon f \rangle \supset \langle h, \epsilon h, \epsilon e', \epsilon l, \epsilon f \rangle \supset 0$. Pragmatics: declare $\deg(h, e', l, f, \epsilon) = (1, 1, 0, 0, 1)$ and set $t := e^h$ and $e := (t - 1)e'/h$.

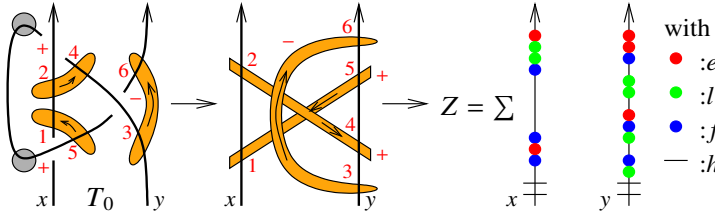
How did it arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^-/\mathfrak{h} =: sl_2^+/\mathfrak{h}$, where $\mathfrak{b}^+ = \langle l, f \rangle/[f, l] = f$ is a Lie bialgebra with $\delta: \mathfrak{b}^+ \rightarrow \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta: (l, f) \mapsto (0, l \wedge f)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle h', e', l, f \rangle/\dots$. **Idea.** Replace $\delta \rightarrow \epsilon\delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 1$, get $[f, l] = f$, $[f, h'] = -\epsilon f$, $[l, e'] = e'$, $[h', e'] = -\epsilon e'$, $[h', l] = 0$, and $[e', f] = h' - \epsilon l$. Now note that $h' + \epsilon l$ is central, so switch to $h := h' + \epsilon l$. This is \mathfrak{g}_1 .

Ordering Symbols. \odot (*poly* | *specs*) plants the variables of *poly* in $\hat{S}(\oplus_{\mathfrak{g}})$ along $\hat{U}(\mathfrak{g})$ according to *specs*. E.g.,

$$\odot(e_1 e^{\epsilon^3} l_1^3 l_2 f_3^9 | f_3 l_1 e_1 e_3 l_2) = f^9 l^3 e e^{\epsilon} l \in \hat{U}(\mathfrak{g}).$$

This enables the description of elements of $\hat{U}(\mathfrak{g})$ using commutative polynomials / power series. In \mathfrak{g}_1 , no need to specify h/t .

Algebras and Invariants. Given any unital algebra A (even better if A is Hopf; typically, $A \sim \hat{U}(\mathfrak{g})$), appropriate **orange** $R \in A \otimes A$, and appropriate cuaps $\in A$, get an $A^{\otimes S}$ -valued invariant of pure S -component tangles:



What we didn't say (more, including videos, in $\omega\epsilon\beta$ /Talks).

- ρ_1 is “line” in the coloured Jones polynomial; related to Melvin-Morton-Rozansky.
- ρ_1 extends to “rotational virtual tangles” and is a projection of the universal finite type invariant of such.
- ρ_1 seems to have a better chance than anything else we know to detect a counterexample to slice=ribbon.
- ρ_1 leads to many questions and a very long to-do list. Years of work, many papers ahead. Have fun!

Demo Programs.

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 $\omega\epsilon\beta$ /Demo
CF [E_] := Module[{vars = Union@Cases[E, e_ | 1_ | f_, \infty]},
  If[vars === {}, Factor[E],
    Total[CoefficientRules[E, vars] /.
      (p_ -> c_) => Factor[c] Times @@ (vars^p) ] ];
CF [E_#] := CF /@ E;
E [i_, j_, s_] := E [1, (-1)^s 1_j, (-t)^s e_i f_j,
  t^s e_i 1_{(1+s) i-s j} f_j + (-1)^s 1_i 1_j + (-t^2)^s e_i^2 f_j^2 / 4];
E [i_, s_] := E [1, \theta, \theta, s 1_i];
E /: E [1, L1_, Q1_, P1_] E [1, L2_, Q2_, P2_] :=
  E [1, L1 + L2, Q1 + Q2, P1 + P2];

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Formatting
(prints differ ☺)

Preparation

z1 = ($\mathbb{E}[1, 11, 0]$ $\mathbb{E}[4, 2, -1]$ $\mathbb{E}[15, 5, 0]$ **Preparing the Trefoil**
 $\mathbb{E}[6, 8, -1]$ $\mathbb{E}[9, 16, 0]$ $\mathbb{E}[12, 14, -1]$ $\mathbb{E}[3, -1]$ $\mathbb{E}[7, +1]$
 $\mathbb{E}[10, -1]$ $\mathbb{E}[13, +1]$)

$$\mathbb{E} \left[1, -l_2 + l_5 - l_8 + l_{11} - l_{14} + l_{16}, \right. \\ \left. - \frac{e_4 f_2}{t} + e_{15} f_5 - \frac{e_6 f_8}{t} + e_1 f_{11} - \frac{e_{12} f_{14}}{t} + e_9 f_{16}, \right. \\ \left. - \frac{e_2^2 f_2^2}{4 t^2} + \frac{1}{4} e_{15}^2 f_5^2 - \frac{e_6^2 f_8^2}{4 t^2} + \frac{1}{4} e_1^2 f_{11}^2 - \frac{e_{12}^2 f_{14}^2}{4 t^2} + \frac{1}{4} e_9^2 f_{16}^2 + e_1 f_{11} l_1 + \right. \\ \left. \frac{e_4 f_2 l_2}{t} - l_3 - l_2 l_4 + l_7 + \frac{e_6 f_8 l_8}{t} - l_6 l_8 + e_9 f_{16} l_9 - l_{10} + \right. \\ \left. l_1 l_{11} + l_{13} + \frac{e_{12} f_{14} l_{14}}{t} - l_{12} l_{14} + e_{15} f_5 l_{15} + l_5 l_{15} + l_9 l_{16} \right]$$

DP_{x₁→D_α, y₁→D_β} [P_] [f_] := **Differential Polynomials**
Total[CoefficientRules[P, {x, y}] /. (Implementing $P(\partial_\alpha, \partial_\beta)(f)$)
 ({m_, n_} -> c_) => c D[f, {α, m}, {β, n}]]

S_{1_j} (x:e|f)_i → R_- [E [ω_-, L_-, Q_-, P_-]] := **le and fl Sorts**
 With [{λ = ∂_{1_j} L, α = ∂_{x_i} Q, q = e^γ β x_k + γ 1_k }, CF [E [ω, L /. 1_j → 1_k, t^λ α x_k + (Q /. x_i → θ), e^{-q} DP_{1_j→D_α, x_i→D_β} [P] [e^q] /. {β → α / ω, γ → λ Log[t]}]]];

$$\Delta [k_-] := ((t - 1) (2 (\alpha \beta + \delta \mu)^2 - \alpha^2 \beta^2) - 4 e_k 1_k f_k \delta^2 \mu^2 - \delta (1 + \mu) (f_k^2 \alpha^2 + e_k^2 \beta^2) - e_k^2 f_k^2 \delta^3 (1 + 3 \mu) - \text{The Λόγος} \\ 2 (\alpha \beta + 2 \delta \mu + e_k f_k \delta^2 (1 + 2 \mu) + 2 1_k \delta \mu^2) (f_k \alpha + e_k \beta) - 4 (1_k \mu^2 + e_k f_k \delta (1 + \mu)) (\alpha \beta + \delta \mu) (1 + t) / 4;$$

S_{f_i e_j → R_-} [E [ω_-, L_-, Q_-, P_-]] := **fe Sorts**
 With [{q = ((1 - t) α β + β e_k + α f_k + δ e_k f_k) / μ }, CF [E [μ ω, L, μ ω q + μ (Q /. f_i | e_j → θ), μ⁴ e^{-q} DP_{f_i→D_α, e_j→D_β} [P] [e^q] + ω^k Δ [k]] /. μ → 1 + (t - 1) δ / . {α → ω⁻¹ (∂_{f_i} Q /. e_j → θ), β → ω⁻¹ (∂_{e_j} Q /. f_i → θ), δ → ω⁻¹ ∂_{f_i, e_j} Q}]];

m_{i_-, j_- → R_-} [Z_#] := **Module** [{x, z}, **CF** [(Z // S_{f_i e_j → x} // S_{1_i e_x → x} // S_{f_x 1_j → x}) /. z_{-i|j|x} → z_k]] **Elf Merges**

(Do [z1 = z1 // m_{1, k+1}, {k, 2, 16}]; z1) **Rewriting the Trefoil**
 (by merging 16 elves)
 $\mathbb{E} \left[\frac{1-t+t^2}{t}, \theta, \theta, \frac{(-1+t) (1-t+t^2)^2 (1-t+2t^2)}{t^3} - \frac{2 (1+t) (1-t+t^2)^3 e_1 f_1}{t^4} - \frac{2 (-1+t) (1+t) (1-t+t^2)^3 1_1}{t^4} \right]$

$$\rho_1 [\mathbb{E} [\omega_-, _, _, P_-]] := \text{CF} \left[\frac{t ((P /. e_- | 1_- | f_- \rightarrow \theta) - t \omega^3 (\partial_t \omega))}{(t - 1)^2 \omega^2} \right]$$

ρ_1 [z1] // **Expand** **$\rho_1(3_1)$**

References.

[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial*, University of North Carolina PhD thesis, $\omega\epsilon\beta$ /Ov.
 [Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I*, Comm. Math. Phys. **175-2** (1996) 275–296, arXiv:hep-th/9401061.
 [Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial*, Adv. Math. **134-1** (1998) 1–31, arXiv:q-alg/9604005.
 [Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture*, arXiv:math/0201139.

diagram	n_k^a	Alexander's ω^+	genus / ribbon	diagram	n_k^a	Alexander's ω^+	genus / ribbon
		Today's / Rozansky's ρ_1^+	unknotting number / amphicheiral			Today's / Rozansky's ρ_1^+	unknotting number / amphicheiral
	0 ₁ ^a	1	0 / ✓		3 ₁ ^a	t - 1	1 / ✗
	0		0 / ✓		t		1 / ✗
	4 ₁ ^a	3 - t	1 / ✗		5 ₁ ^a	t ² - t + 1	2 / ✗
	0		1 / ✓		2t ³ + 3t		2 / ✗