

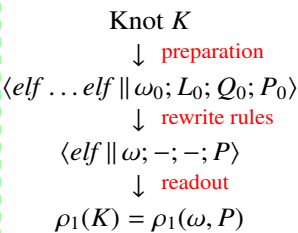


# On Elves and Invariants

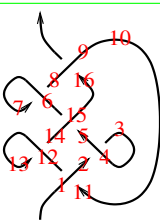
**Abstract.** Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

**Three steps** to the computation of  $\rho_1$ :

- 1. Preparation.** Given  $K$ , results  $\langle \text{long word} \parallel \text{simple formulas} \rangle$ .
- 2. Rewrite rules.** Make the word simpler and the formulas more complicated, until the word "elf" is reached.
- 3. Readout.** The invariant  $\rho_1$  is read from the last formulas.



**Preparation.** Draw  $K$  using a 0-framed 0-rotation planar diagram  $D$  where all crossings are pointing up. Walk along  $D$  labeling features by  $1, \dots, m$  in order: over-passes, under-passes, and right-heading cups and caps ("±-cuaps"). If  $x$  is a xing, let  $i_x$  and  $j_x$  be the labels on its over/under strands, and let  $s_x$  be 0 if it right-handed and  $-1$  otherwise. If  $c$  is a cuap, let  $i_c$  be its label and  $s_c$  be its sign. Set



$$(L; Q; P) = \sum_{x: (i,j,s)} (-)^s \left( l_j; t^s e_i f_j; (-t)^s e_i l_{(1+s)i-sj} f_j + l_i l_j + \frac{t^{2s} e_i^2 f_j^2}{4} \right) + \sum_{c: (i,s)} (0; 0; s \cdot l_i).$$

This done, output  $\langle e_1 l_1 f_1 e_2 l_2 f_2 \dots e_m l_m f_m \parallel 1; L; Q; P \rangle$ .

**In formulas.**  $L$  is always  $\mathbb{Z}$ -linear in  $\{l_i\}$ ,  $Q$  is an  $R$ -linear combination of  $\{e_i f_j\}$  where  $R := \mathbb{Q}[t^{\pm 1}]$ , and  $P$  is an  $R$ -linear combination of  $\{1, l_i, l_i l_j, e_i f_j, e_i l_j f_k, e_i e_j f_k f_l\}$ . (The key to computability!)

**Rewrite Rules.** Manipulate  $\langle \text{word} \parallel \text{formulas} \rangle$  expressions using the rewrite rules below, until you come to the form  $\langle e_1 l_1 f_1 \parallel \omega; -; -; P \rangle$ . Output  $(\omega, P)$ .

**Rule 1, Deletions.** If a letter appears in word but not in formulas, you can delete it.

**Rule 2, Merges.** In word, you can replace adjacent  $v_i v_j$  with  $v_k$  (for  $v \in \{e, l, f\}$ ) while making the same changes in formulas (provided  $k$  creates no naming clashes). E.g.,

$$\langle \dots e_i e_j \dots \parallel Z \rangle \rightarrow \langle \dots e_k \dots \parallel Z|_{e_i, e_j \rightarrow e_k} \rangle.$$

**Rule 3, le Sorts.** Provided  $k$  introduces no clashes, given  $\langle \dots l_j e_i \dots \parallel \omega; L; Q; P \rangle$ , decompose  $L = \lambda l_j + L'$ ,  $Q = \alpha e_i + Q'$ , write  $P = P(e_i, l_j)$  (with messy coefficients), set  $q = e^\gamma \beta e_k + \gamma l_k$ , and output

$$\langle \dots e_k l_k \dots \parallel \omega; L|_{l_j \rightarrow l_k}; t^\lambda \alpha e_k + Q'; e^{-q} P(\partial_\beta, \partial_\gamma) e^q |_{\beta \rightarrow \alpha/\omega, \gamma \rightarrow \lambda \log t} \rangle.$$

**Rule 4, fl Sorts.** Provided  $k$  introduces no clashes, given  $\langle \dots f_i l_j \dots \parallel \omega; L; Q; P \rangle$ , decompose  $L = \lambda l_j + L'$ ,  $Q = \alpha f_i + Q'$ , write  $P = P(f_i, l_j)$  (with messy coefficients), set  $q = e^\gamma \beta f_k + \gamma l_k$ , and output

$$\langle \dots l_k f_k \dots \parallel \omega; L|_{l_j \rightarrow l_k}; t^\lambda \alpha f_k + Q'; e^{-q} P(\partial_\beta, \partial_\gamma) e^q |_{\beta \rightarrow \alpha/\omega, \gamma \rightarrow \lambda \log t} \rangle.$$



Happy Birthday, Scott!



"God created the knots, all else in topology is the work of mortals."



www.katlas.org The Knot Atlas Project

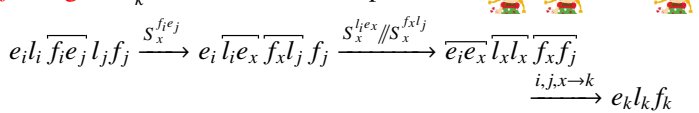
**Rule 5, fe Sorts.** Provided  $k$  introduces no clashes, given  $\langle \dots f_i e_j \dots \parallel \omega; L; Q; P \rangle$ , decompose  $Q = Q_{fe} f_i e_j + Q_{fj} f_i + Q_{ee} e_j + Q'$  write  $P = P(f_i, e_j)$  (with messy coefficients), set  $\mu = 1 + (t-1)\delta$  and  $q = ((1-t)\alpha\beta + \beta e_k + \alpha f_k + \delta e_k f_k)/\mu$ , and output

$$\left\langle \dots e_k f_k \dots \parallel \begin{matrix} \mu\omega; L; \mu\omega q + \mu Q'; \\ \omega^4 \Lambda_k + e^{-q} P(\partial_\alpha, \partial_\beta)(e^q) \end{matrix} \right\rangle \Big|_{\substack{\alpha \rightarrow Q_{fj}/\omega, \beta \rightarrow Q_{ee}/\omega, \\ \delta \rightarrow Q_{fe}/\omega}}$$

where  $\Lambda_k$  is the Λόγος, "a principle of order and knowledge":

$$\Lambda_k = \frac{t+1}{4} \left( -\delta(\mu+1)(\beta^2 e_k^2 + \alpha^2 f_k^2) - \delta^3(3\mu+1)e_k^2 f_k^2 - 2(\beta e_k + \alpha f_k)(\alpha\beta + 2\delta\mu + \delta^2(2\mu+1)e_k f_k + 2\delta\mu^2 l_k) - 4(\alpha\beta + \delta\mu)(\delta(\mu+1)e_k f_k + \mu^2 l_k) - 4\delta^2 \mu^2 e_k f_k l_k + (t-1)(2(\alpha\beta + \delta\mu)^2 - \alpha^2 \beta^2) \right).$$

**elf merges,**  $m_k^{ij}$ , are defined as compositions



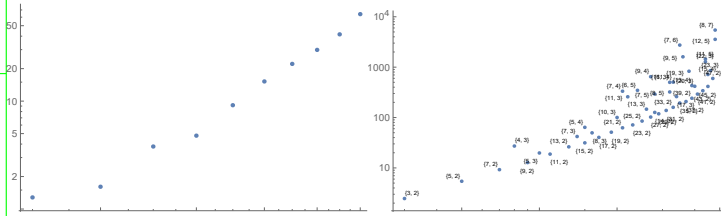
**Readout.** Given  $\langle \text{elf} \parallel \omega; -; -; P \rangle$ , output

$$\rho_1(K) := \frac{t(P|_{e,l,f \rightarrow 0} - t\omega^3)}{(t-1)^2 \omega^2}.$$



( $\omega$  is the Alexander polynomial,  $L$  and  $Q$  are not interesting).

**Experimental Analysis (ωεβ/Exp).** Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



**Power.** On the 250 knots with at most 10 crossings, the pair  $(\omega, \rho_1)$  attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

**Genus.** Up to 12 xings, always  $\rho_1$  is symmetric under  $t \leftrightarrow t^{-1}$ . With  $\rho_1^+$  denoting the positive-degree part of  $\rho_1$ , always  $\deg \rho_1^+ \leq 2g - 1$ , where  $g$  is the 3-genus of  $K$  (equality for 2530 knots). This gives a lower bound on  $g$  in terms of  $\rho_1$  (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

**Why Works?** The Lie algebra  $\mathfrak{g}_1$  (below) is a "solvable approximation of  $\mathfrak{sl}_2$ ".

**Theorem.** The map (as defined below)

$\langle w \parallel \omega; L; Q; P \rangle \mapsto \mathbb{O} \left( \omega^{-1} e^{L \log t + \omega^{-1} Q} (1 + \epsilon \omega^{-4} P) : w \right) \in \hat{\mathcal{U}}(\mathfrak{g}_1)$

is well defined modulo the sorting rules. It maps the initial preparation to a product of "R-matrices" and "cuap values" satisfying the usual moves for Morse knots (R3, etc.). (And hence the result is a "quantum invariant", except computed very differently; no representation theory!).