

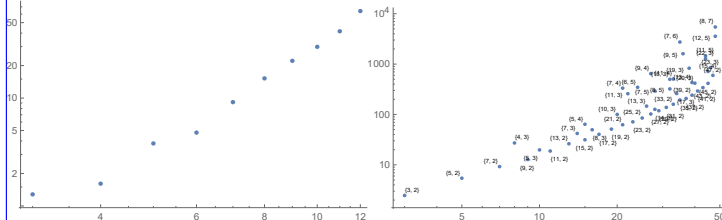


# The Dogma is Wrong

**Abstract.** It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: “quantize and use representation theory”. We present an alternative and better procedure: “centrally extend, approximate by solvable, and learn how to re-order exponentials in a universal enveloping algebra”. While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

**KiW 43 Abstract** ([oebf/kiw](#)). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

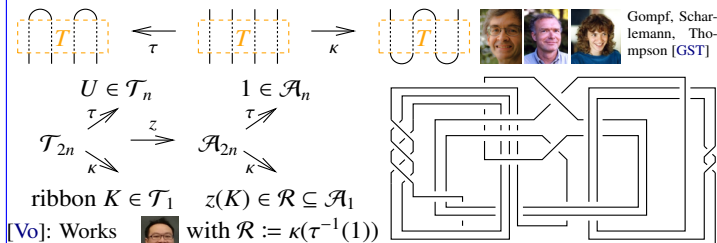
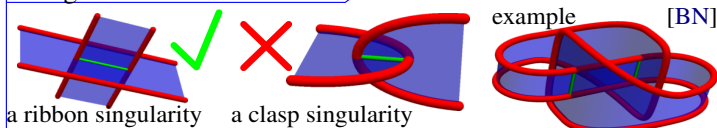
**Experimental Analysis** ([oebf/Exp](#)). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



**Power.** On the 250 knots with at most 10 crossings, the pair  $(\omega, \rho_1)$  attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

**Genus.** Up to 12 crossings, always  $\rho_1$  is symmetric under  $t \leftrightarrow t^{-1}$ . With  $\rho_1^+$  denoting the positive-degree part of  $\rho_1$ , always  $\deg \rho_1^+ \leq 2g - 1$ , where  $g$  is the 3-genus of  $K$  (equality for 2530 knots). This gives a lower bound on  $g$  in terms of  $\rho_1$  (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-tring Alexander failures it does give the right answer.

### Ribbon Knots.



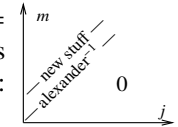
[Vo]: Works for Alexander!  $A^+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$   
 $\rho_1^+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 +$   
Faster is better, leaner is meaner!  $108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$

**dog·ma** (dɔg'mə, dɔg'-' ) The Free Dictionary, [oebf/TFD](#)  
n. pl. **dog-mas** or **dog-ma-ta** (-mə-tə)  
1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.  
2. A principle or statement of ideas, or a group of such principles or statements especially when considered to be authoritative or accepted uncritically: “Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry” (Bertrand Russell).

**Theorem** ([BNG], conjectured [MM], elucidated [Ro1]). Let  $J_d(K)$  be the coloured Jones polynomial of  $K$ , in the  $d$ -dimensional representation of  $sl_2$ . Writing

$$\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}} \Big|_{q=e^h} = \sum_{j,m \geq 0} a_{jm}(K) d^j h^m,$$

“below diagonal” coefficients vanish,  $a_{jm}(K) = 0$  if  $j > m$ , and “on diagonal” coefficients give the inverse of the Alexander polynomial:  $(\sum_{m=0}^{\infty} a_{mm}(K) h^m) \cdot \omega(K)(e^h) = 1$ .



“Above diagonal” we have **Rozansky’s Theorem** [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})\omega(K)(q^d)} \left( 1 + \sum_{k=1}^{\infty} \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)} \right).$$

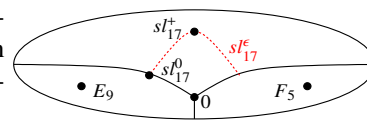
**The Yang-Baxter Technique.** Given an algebra  $A$  (typically  $\hat{\mathcal{U}}(\mathfrak{g})$  or  $\hat{\mathcal{U}}_q(\mathfrak{g})$ ) and elements  $R = \sum a_i \otimes b_i \in A \otimes A$  and  $C \in A$ , form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

**Problem.** Extract information from  $Z$ .  
**The Dogma.** Use representation theory. In principle finite, but slow.

**The Loyal Opposition.** For certain algebras, work in a homomorphic poly-dimensional “space of formulas”.  $m_k^{ij} \circlearrowleft \{ \mathcal{F}_S \} \xrightarrow{\mathbb{E}} \{ A^{\otimes S} \} \circlearrowright m_k^{ij}$

**The (fake) moduli** of Lie algebras on  $V$ , a quadratic variety in  $(V^*)^{\otimes 2} \otimes V$  is on the right. We care about  $sl_{17}^k := sl_{17}^e / (\epsilon^{k+1} = 0)$ .



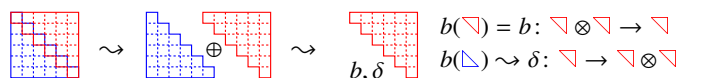
**Why are “solvable algebras” any good?** Contrary to common beliefs, computations in semi-simple Lie algebras are just awful:

```
ln[1]= MatrixExp [ { a b } ] // FullSimplify // MatrixForm Enter
```

Yet in solvable algebras, exponentiation is fine and even BCH,  $z = \log(e^x e^y)$ , is bearable:

```
ln[2]= MatrixExp [ { a b } ] // MatrixForm Out[2]/MatrixForm= { e^a b (e^a - e^c) }
ln[3]= MatrixExp [ { a1 b1 } ] . MatrixExp [ { a2 b2 } ] // MatrixForm
MatrixLog // PowerExpand // Simplify // Enter
```

**Recomposing  $gl_n$ .** Half is enough!  $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(\nabla, b, \delta)$ :



Now define  $gl_n^e := \mathcal{D}(\nabla, b, \epsilon\delta)$ . Schematically, this is  $[\nabla, \nabla] = \nabla$ ,  $[\nabla, \Delta] = \epsilon\Delta$ , and  $[\nabla, \Delta] = \Delta + \epsilon\nabla$ . In detail, it is

$$\begin{matrix} & i & j \\ i & \begin{matrix} e_{ij} \\ h_i \end{matrix} & \\ j & \begin{matrix} f_{ji} \\ g_j \end{matrix} & \end{matrix} \quad \begin{matrix} [e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj} & [f_{ij}, f_{kl}] = \epsilon \delta_{jk} f_{il} - \epsilon \delta_{il} f_{kj} \\ [e_{ij}, f_{kl}] = \delta_{jk} (\epsilon \delta_{i < k} e_{il} + \delta_{il} (h_i + \epsilon g_i) / 2 + \delta_{i > l} f_{il}) \\ & & - \delta_{il} (\epsilon \delta_{k < j} e_{kj} + \delta_{kj} (h_j + \epsilon g_j) / 2 + \delta_{k > j} f_{kj}) \\ [g_i, e_{jk}] = (\delta_{ij} - \delta_{ik}) e_{jk} & [h_i, e_{jk}] = \epsilon (\delta_{ij} - \delta_{ik}) e_{jk} \\ [g_i, f_{jk}] = (\delta_{ij} - \delta_{ik}) f_{jk} & [h_i, f_{jk}] = \epsilon (\delta_{ij} - \delta_{ik}) f_{jk} \end{matrix}$$