

**The PBW Problem.** In  $\mathcal{U}(\mathfrak{g}^\epsilon)$ , bring  $Z = y^3 a^2 x^2 \cdot y^2 a^2 x$  to  $yax$ -order. In other words, find  $g \in \mathbb{Z}[\epsilon, t, y, a, x]$  such that  $Z = \mathbb{O}(f = y^3 y_2^2 a_1^2 a_2^2 x_1^2 x_2 : y_1 a_1 x_1 y_2 a_2 x_2) = \mathbb{O}(g : yax)$ .

**Solution, Part 1.** In  $\mathcal{U}(\mathfrak{g}^\epsilon)$  we have

$$X_{\tau_1, \eta_1, \alpha_1, \xi_1, \tau_2, \eta_2, \alpha_2, \xi_2} := e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x} e^{\tau_2 t} e^{\eta_2 y} e^{\alpha_2 a} e^{\xi_2 x} = e^{\tau t} e^{\eta y} e^{\alpha a} e^{\xi x} =: Y_{\tau, \eta, \alpha, \xi},$$

where  $\tau, \eta, \alpha, \xi$  are ugly functions of  $\tau_i, \eta_i, \alpha_i, \xi_i$ :

$$\begin{aligned} \tau &= \tau_1 + \tau_2 - \frac{\log(1 - \epsilon \eta_2 \xi_1)}{\epsilon} = \tau_1 + \tau_2 + \eta_2 \xi_1 + \frac{\epsilon}{2} \eta_2^2 \xi_1^2 + \dots, \\ \eta &= \eta_1 + \frac{e^{-\alpha_1} \eta_2}{(1 - \epsilon \eta_2 \xi_1)} = \eta_1 + e^{-\alpha_1} \eta_2 + \epsilon e^{-\alpha_1} \eta_2^2 \xi_1 + \dots, \\ \alpha &= \alpha_1 + \alpha_2 + 2 \log(1 - \epsilon \eta_2 \xi_1) = \alpha_1 + \alpha_2 - 2 \epsilon \eta_2 \xi_1 + \dots, \\ \xi &= \frac{e^{-\alpha_2} \xi_1}{(1 - \epsilon \eta_2 \xi_1)} + \xi_2 = e^{-\alpha_2} \xi_1 + \xi_2 + \epsilon e^{-\alpha_2} \eta_2 \xi_1^2 + \dots \end{aligned}$$

**Note 1.** This defines a mapping  $\Phi: \mathbb{R}_{\tau_1, \eta_1, \alpha_1, \xi_1, \tau_2, \eta_2, \alpha_2, \xi_2}^8 \rightarrow \mathbb{R}_{\tau, \eta, \alpha, \xi}^4$ .

**Proof.**  $\mathfrak{g}^\epsilon$  has a 2D representation  $\rho$ :

$$\begin{aligned} \rho t &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \rho y = \begin{pmatrix} 0 & \theta \\ -\epsilon & 0 \end{pmatrix}; \\ \rho a &= \begin{pmatrix} (1 + 1/\epsilon) / 2 & 0 \\ 0 & -(1 - 1/\epsilon) / 2 \end{pmatrix}; \quad \rho x = \begin{pmatrix} 0 & 1 \\ 0 & \theta \end{pmatrix}; \end{aligned}$$

$$\begin{aligned} \text{Simplify} \{ \rho a \cdot \rho x - \rho x \cdot \rho a = \rho x, \quad \rho a \cdot \rho y - \rho y \cdot \rho a = -\rho y, \\ \rho x \cdot \rho y - \rho y \cdot \rho x = \rho t - 2 \epsilon \rho a \} \end{aligned}$$

{True, True, True}

It is enough to verify the desired identity in  $\rho$ :

ME = MatrixExp;

Simplify [

$$\begin{aligned} & \text{ME}[\tau_1 \rho t] \cdot \text{ME}[\eta_1 \rho y] \cdot \text{ME}[\alpha_1 \rho a] \cdot \text{ME}[\xi_1 \rho x] \cdot \text{ME}[\tau_2 \rho t] \cdot \\ & \text{ME}[\eta_2 \rho y] \cdot \text{ME}[\alpha_2 \rho a] \cdot \text{ME}[\xi_2 \rho x] = \\ & \text{ME}[\tau_0 \rho t] \cdot \text{ME}[\eta_0 \rho y] \cdot \text{ME}[\alpha_0 \rho a] \cdot \text{ME}[\xi_0 \rho x] / . \\ & \left\{ \begin{aligned} \tau_0 &\rightarrow -\frac{\log[1 - \epsilon \eta_2 \xi_1]}{\epsilon} + \tau_1 + \tau_2, \quad \eta_0 \rightarrow \eta_1 + \frac{e^{-\alpha_1} \eta_2}{1 - \epsilon \eta_2 \xi_1}, \\ \alpha_0 &\rightarrow 2 \text{Log}[1 - \epsilon \eta_2 \xi_1] + \alpha_1 + \alpha_2, \quad \xi_0 \rightarrow \frac{e^{-\alpha_2} \xi_1}{1 - \epsilon \eta_2 \xi_1} + \xi_2 \end{aligned} \right\} \end{aligned}$$

True

**Solution, Part 2.** But now, with  $D_f = f(z \mapsto \partial_z) = \partial_{\eta_1}^3 \partial_{\alpha_1}^2 \partial_{\xi_1}^2 \partial_{\eta_2}^2 \partial_{\alpha_2}^2 \partial_{\xi_2}$ ,

$$\begin{aligned} Z &= D_f X_{\tau_1, \eta_1, \alpha_1, \xi_1, \tau_2, \eta_2, \alpha_2, \xi_2} \Big|_{v_S=0} = D_f Y_{\tau, \eta, \alpha, \xi} \Big|_{v_S=0} \\ &= \mathbb{O} \left( D_f e^{\tau t} e^{\eta y} e^{\alpha a} e^{\xi x} \Big|_{v_S=0} : yax \right) = \mathbb{O}(g : yax) : \end{aligned}$$

$$\begin{aligned} \text{Expand} \left[ \partial_{(\eta_1, 3)} \partial_{(\alpha_1, 2)} \partial_{(\xi_1, 2)} \partial_{(\eta_2, 2)} \partial_{(\alpha_2, 2)} \partial_{(\xi_2, 1)} \text{Exp} \left[ \right. \right. \\ \left. \left. \left( -\frac{\log[1 - \epsilon \eta_2 \xi_1]}{\epsilon} + \tau_1 + \tau_2 \right) t + \left( \eta_1 + \frac{e^{-\alpha_1} \eta_2}{1 - \epsilon \eta_2 \xi_1} \right) y + \right. \right. \\ \left. \left. (2 \text{Log}[1 - \epsilon \eta_2 \xi_1] + \alpha_1 + \alpha_2) a + \left( \frac{e^{-\alpha_2} \xi_1}{1 - \epsilon \eta_2 \xi_1} + \xi_2 \right) x \right. \right. \\ \left. \left. \right] / . (\tau | \eta | \alpha | \xi)_{1|2} \rightarrow \theta \right] \end{aligned}$$

$$\begin{aligned} & 2 a^4 t^2 x y^3 + 4 t x^2 y^4 - 16 a t x^2 y^4 + 24 a^2 t x^2 y^4 - 16 a^3 t x^2 y^4 + \\ & 4 a^4 t x^2 y^4 + 16 x^3 y^5 - 32 a x^3 y^5 + 24 a^2 x^3 y^5 - 8 a^3 x^3 y^5 + a^4 x^3 y^5 + \\ & 2 a^4 t x y^3 \epsilon - 8 a^5 t x y^3 \epsilon + 8 x^2 y^4 \epsilon - 40 a x^2 y^4 \epsilon + 80 a^2 x^2 y^4 \epsilon - \\ & 80 a^3 x^2 y^4 \epsilon + 40 a^4 x^2 y^4 \epsilon - 8 a^5 x^2 y^4 \epsilon - 4 a^5 x y^3 \epsilon^2 + 8 a^6 x y^3 \epsilon^2 \end{aligned}$$

diagram	$n_k^t$ Alexander's $\omega^+$ Today's / Rozansky's $\rho_1^+$	genus / ribbon unknotting number / amphicheiral	diagram	$n_k^t$ Alexander's $\omega^+$ Today's / Rozansky's $\rho_1^+$	genus / ribbon unknotting number / amphicheiral
	$0_1^a$ 0	1 0/✓		$3_1^a$ $t$	$t-1$ 1/✗ 1/✗
	$4_1^a$ 0	$3-t$ 1/✗ 1/✓		$5_1^a$ $2t^3 + 3t$	$t^2 - t + 1$ 2/✗ 2/✗
	$5_2^a$ $5t-4$	$2t-3$ 1/✗ 1/✗		$6_1^a$ $t-4$	$5-2t$ 1/✓ 1/✗

**Note 2.** Replacing  $f \rightarrow D_f$  (and likewise  $g \rightarrow D_g$ ), we find that  $D_g = \Phi_* D_f$ .

**Note 3.** The two great evils of mathematics are non-commutativity and non-linearity. We traded one for the other.

**Note 4.** We could have done similarly with  $e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x} = e^{\tau t + \eta y + \alpha a + \xi x}$ , and with  $S(e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x})$ ,  $\Delta(e^{\tau_1 t} e^{\eta_1 y} e^{\alpha_1 a} e^{\xi_1 x})$ ,  $\prod_{i=1}^5 e^{\tau_i t} e^{\eta_i y} e^{\alpha_i a} e^{\xi_i x}$ .

**Fact.**  $R_{12} \rightarrow \exp(\partial_{\tau_1} \partial_{\alpha_2} + \partial_{y_1} \partial_{x_2})(1 + \sum_{d \geq 1} \epsilon^d p_d)$ , where the  $p_d$  are computable polynomials of a-priori bounded degrees.

**Moral.** We need to understand the pushforwards via maps like  $\Phi$  of (formally  $\infty$ -order) “differential operators at 0”, that in themselves are perturbed Gaussians. This turns out to be the same problem as “0-dimensional QFT” (except no integration is ever needed), and if  $\epsilon^{k+1} = 0$ , it is explicitly soluble.

**References.**

[BN] D. Bar-Natan, *Polynomial Time Knot Polynomial*, research proposal for the 2017 Killam Fellowship, [oeqf/K17](#).  
[BNG] D. Bar-Natan and S. Garoufalidis, *On the Melvin-Morton-Rozansky conjecture*, *Invent. Math.* **125** (1996) 103–133.  
[BV1] D. Bar-Natan and R. van der Veen, *A Polynomial Time Knot Polynomial*, [arXiv:1708.04853](#).  
[BV2] D. Bar-Natan and R. van der Veen, *Poly-Time Knot Polynomials Via Solvable Approximations*, in preparation.  
[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures*, *Geom. and Top.* **14** (2010) 2305–2347, [arXiv:1103.1601](#).  
[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function*, *Commun. Math. Phys.* **169** (1995) 501–520.  
[Ov] A. Overbay, *Perturbative Expansion of the Colored Jones Polynomial*, University of North Carolina PhD thesis, [oeqf/Ov](#).  
[Ro1] L. Rozansky, *A contribution of the trivial flat connection to the Jones polynomial and Witten's invariant of 3d manifolds, I*, *Comm. Math. Phys.* **175-2** (1996) 275–296, [arXiv:hep-th/9401061](#).  
[Ro2] L. Rozansky, *The Universal R-Matrix, Burau Representation and the Melvin-Morton Expansion of the Colored Jones Polynomial*, *Adv. Math.* **134-1** (1998) 1–31, [arXiv:q-alg/9604005](#).  
[Ro3] L. Rozansky, *A Universal U(1)-RCC Invariant of Links and Rationality Conjecture*, [arXiv:math/0201139](#).  
[Vo] H. Vo, University of Toronto Ph.D. thesis, in preparation.

dog·ma (dōg'mə, dōg'-)

The Free Dictionary, [oeqf/TFD](#)

*n. pl. dog-mas or dog-ma-ta* (-mə-tə)

1. A doctrine or a corpus of doctrines relating to matters such as morality and faith, set forth in an authoritative manner by a religion.
2. A principle or statement of ideas, or a group of such principles or statements, especially when considered to be authoritative or accepted uncritically: “*Much education consists in the instilling of unfounded dogmas in place of a spirit of inquiry*” (Bertrand Russell).