2. The "$z_i \to z_j$ variable rename map $\sigma^i_j\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j)$ becomes ${}^t\sigma^i_j = \mathbb{e}^{z_j\zeta_i}$, and it's easy to rename several variables simultaneously.
3. The "archetypal multiplication map $m^{ij}_k\colon \mathcal{S}(z_i, z_j) \to \mathcal{S}(z_k)$" has ${}^tm = \mathbb{e}^{z_k(\zeta_i+\zeta_j)}$.
4. The "archetypal coproduct $\Delta^i_{jk}\colon \mathcal{S}(z_i) \to \mathcal{S}(z_j, z_k)$", given by $z_i \to z_j + z_k$ or $\Delta z = z \otimes 1 + 1 \otimes z$, has ${}^t\Delta = \mathbb{e}^{(z_j+z_k)\zeta_i}$.
5. R-matrices tend to have terms of the form $\mathbb{e}_q^{\hbar y_1 x_2} \in \mathcal{U}_q \otimes \mathcal{U}_q$. The "baby R-matrix" is ${}^tR = \mathbb{e}^{\hbar yx} \in \mathcal{S}(y, x)$.

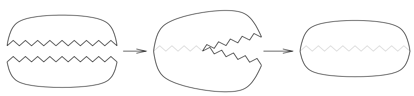
Proposition. If $F\colon \mathcal{S}(B) \to \mathcal{S}(B')$ is linear and "continuous", then ${}^tF = \exp\left(\sum_{z_i\in B} \zeta_i z_i\right) /\!\!/ F$.

The Heisenberg Example. The "Weyl form of the canonical commutation relations" states that if $[y, x] = t$ and t is central, then $\mathbb{e}^{\xi x}\mathbb{e}^{\eta y} = \mathbb{e}^{\eta y}\mathbb{e}^{\xi x}\mathbb{e}^{-\eta\xi t}$. Thus with

$$SW_{xy} \circlearrowright \mathcal{S}(t, y, x) \underset{\mathbb{O}_{yx}}{\overset{\mathbb{O}_{xy}}{\rightrightarrows}} \mathcal{U}(t, y, x)$$

we have ${}^tSW_{xy} = \mathbb{e}^{\tau t+\eta y+\xi x-\eta\xi t}$.

The Zipping Issue (between unbound and bound lies half-zipped).



Zipping. If $P(\zeta^j, z_i)$ is a polynomial, or whenever otherwise convergent, set

$$\left\langle P(\zeta^j, z_i)\right\rangle_{(\zeta^j)} = P\left(\partial_{z_j}, z_i\right)\Big|_{z_i=0}.$$

(E.g., if $P = \sum a_{nm}\zeta^n z^m$ then $\langle P\rangle_\zeta = \sum n! a_{nn}$).

The Zipping / Contraction Theorem. If P has a finite ζ-degree and the y's and the q's are "small" then

$$\left\langle P(z_i, \zeta^j)\mathbb{e}^{\eta^i z_i + y_j\zeta^j}\right\rangle_{(\zeta^j)} = \left\langle P(z_i + y_i, \zeta^j)\mathbb{e}^{\eta^i(z_i+y_i)}\right\rangle_{(\zeta^j)},$$

(proof: replace $y_j \to \hbar y_j$ and test at $\hbar = 0$ and at $\partial_\hbar$), and

$$\left\langle P(z_i, \zeta^j)\mathbb{e}^{c+\eta^i z_i + y_j\zeta^j + q^i_j z_i\zeta^j}\right\rangle_{(\zeta^j)} = \det(\tilde{q})\left\langle P(\tilde{q}^k_i(z_k + y_k), \zeta^j)\mathbb{e}^{c+\eta^i\tilde{q}^k_i(z_k+y_k)}\right\rangle_{(\zeta^j)}$$

where $\tilde{q}$ is the inverse matrix of $1 - q$: $(\delta^i_j - q^i_j)\tilde{q}^j_k = \delta^i_k$ (proof: replace $q^i_j \to \hbar q^i_j$ and test at $\hbar = 0$ and at $\partial_\hbar$).

Implementation. ωεβ/ZipBindDemo

```
Kδ /: Kδi_,j_ := If[i === j, 1, 0];
{z*, x*, y*} = {ζ, ξ, η}; {ζ*, ξ*, η*} = {z, x, y};
(u_i_)* := (u*)i;
Zip{}[P_] := P;
Zip{ζ_,ζs___}[P_] :=
 (Expand[P // Zip{ζs}] /. f_. ζ^d_. :> ∂{ζ*,d}f) /. ζ* → 0
Zip{ζ}[(a ζ^6 + ζ + 3) (z^5 e^z + 7 z) + 99 b]
7 + 720 a + 99 b
Zip{ξ,η}[ξ^3 η^3 e^(a x + b y + c x y)]
a^3 b^3 + 9 a^2 b^2 c + 18 a b c^2 + 6 c^3
(* E[Q,P] means e^Q P *)
E /: Zipζs_List@E[Q_, P_] :=
  Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ζs}];
   c = Q /. Alternatives @@ (ζs ⋃ zs) → 0;
   ys = Table[∂ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = Inverse@Table[Kδz,ζ* - ∂z,ζ Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule;
   Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ E[Q2, Det[qt] e^-Q2 Zipζs[e^Q1 (P /. zrule)]] ];
```

```
Eh = E[h Sum[a10 i+j xi ξj, {i,1,3},{j,1,3}], Sum[fi[x1, x2, x3] ξi, {i,1,3}]];
E1 = Eh /. h → 1
E[a11 x1 ξ1 + a21 x2 ξ1 + a31 x3 ξ1 + a12 x1 ξ2 +
  a22 x2 ξ2 + a32 x3 ξ2 + a13 x1 ξ3 + a23 x2 ξ3 + a33 x3 ξ3,
 ξ1 f1[x1, x2, x3] + ξ2 f2[x1, x2, x3] + ξ3 f3[x1, x2, x3]]
Short[lhs = Zip{ξ1,ξ2}@E1, 5]
E[((a13 ((-1 + a22) a31 - a21 a32) + a12 (-a23 a31 + a21 a33) +
      (-1 + a11) (a23 a32 - (-1 + a22) a33)) x3 ξ3) /
   (-1 + a12 a21 - a11 (-1 + a22) + a22),
  (<<17>> + a21 <<1>>) / (-1 + a12 a21 - a11 (-1 + a22) + a22)^2]
lhs == Zip{ξ1}@Zip{ξ2}@E1 == Zip{ξ2}@Zip{ξ1}@E1
True
Short[
 lhs = Normal[Eh /. E[Q_, P_] :> Series[P e^Q, {h, 0, 3}]] //
   Zip{ξ1,ξ2}, 5]
h a13 ξ3 f1[0, 0, x3] + 2 h^2 a11 a13 ξ3 f1[0, 0, x3] +
 3 h^3 a11^2 a13 ξ3 f1[0, 0, x3] + 2 h^3 a12 a13 a21 ξ3 f1[0, 0, x3] +
 h^2 a13 a22 ξ3 f1[0, 0, x3] + <<337>> +
 1/6 h^3 a31^3 x3^3 ξ3 f3^(3,0,0)[0, 0, x3] + 1/2 h^3 a31^2 a32 x3^3 f1^(3,1,0)[0, 0, x3] +
 1/6 h^3 a31^3 x3^3 f2^(3,1,0)[0, 0, x3] + 1/6 h^3 a31^3 x3^3 f1^(4,0,0)[0, 0, x3]
rhs =
  Normal[Zip{ξ1,ξ2}@Eh /. E[Q_, P_] :> Series[P e^Q, {h, 0, 3}]];
Simplify[lhs == rhs]
True
E /: E[Q1_, P1_] E[Q2_, P2_] := E[Q1 + Q2, P1 * P2];
Bindζs_List[L_E, R_E] := Module[{n, hideζs, hidezs},
   hideζs = Table[ζs[[i]] → ζn@i, {i, Length@ζs}];
   hidezs = Table[ζs[[i]]* → zn@i, {i, Length@ζs}];
   Zipζs/.hideζs[(L /. hidezs) (R /. hideζs)] ];
Bind{ξ2}[E[ξ (x1 + x2), 1], E[ξ2 (x2 + x3), 1]]
E[ξ (x1 + x2 + x3), 1]
Bind{ξ2}[E[(ξ2 + ξ3) x2, 1], E[(ξ1 + ξ2) x, 1]]
E[x (ξ1 + ξ2 + ξ3), 1]
```

The 2D Lie Algebra. Clever people know* that if $[a, x] = \gamma x$ then $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$. Ergo with

$$SW_{ax} \circlearrowright \mathcal{S}(a, x) \underset{\mathbb{O}_{xa}}{\overset{\mathbb{O}_{ax}}{\rightrightarrows}} \mathcal{U}(a, x)$$

we have ${}^tSW_{ax} = \mathbb{e}^{\alpha a+\mathbb{e}^{-\gamma\alpha}\xi x}$.

* Indeed $xa = (a - \gamma)x$ thus $xa^n = (a - \gamma)^n x$ thus $x\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha(a-\gamma)}x = \mathbb{e}^{-\gamma\alpha}\mathbb{e}^{\alpha a}x$ thus $x^n\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}(\mathbb{e}^{-\gamma\alpha})^n x^n$ thus $\mathbb{e}^{\xi x}\mathbb{e}^{\alpha a} = \mathbb{e}^{\alpha a}\mathbb{e}^{\mathbb{e}^{-\gamma\alpha}\xi x}$.

The Real Thing. In $QU/(\epsilon^2 = 0)$ over $\mathbb{Q}[\![\hbar]\!]$ using the yax order, $T = \mathbb{e}^{\hbar t}$, $\bar{T} = T^{-1}$, $\mathcal{A} = \mathbb{e}^{\gamma\alpha}$, and $\bar{\mathcal{A}} = \mathcal{A}^{-1}$, we have

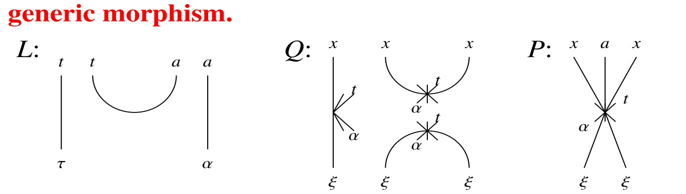
$${}^tR_{ij} = \mathbb{e}^{\hbar(y_i x_j - t_i a_j/\gamma)}\left(1 + \epsilon\hbar\left(a_i a_j/\gamma - \gamma\hbar^2 y_i^2 x_j^2/4\right)\right)$$

in $\mathcal{S}(B_i, B_j)$, and in $\mathcal{S}(B_1^*, B_2^*, B)$ we have

$${}^tm = \mathbb{e}^{(\alpha_1+\alpha_2)a+\eta_2\xi_1(1-T)/\hbar+(\xi_1\bar{\mathcal{A}}_2+\xi_2)x+(\eta_1+\eta_2\bar{\mathcal{A}}_1)y}(1+\epsilon\lambda_m),$$

where $\lambda_m = 2a\eta_2\xi_1 T + \frac{1}{4}\gamma\eta_2^2\xi_1^2\left(3T^2-4T+1\right)/\hbar - \frac{1}{2}\gamma\eta_2\xi_1^2(3T-1)x\bar{\mathcal{A}}_2 - \frac{1}{2}\gamma\eta_2^2\xi_1(3T-1)y\bar{\mathcal{A}}_1 + \gamma\eta_2\xi_1 xy\hbar\bar{\mathcal{A}}_1\bar{\mathcal{A}}_2$. Similar formulas delight us for ${}^t\Delta$ and ${}^tS$.

A generic morphism.



Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/