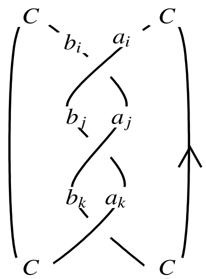


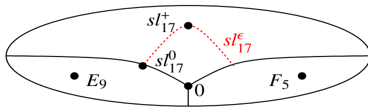

**The Yang-Baxter Technique.** Given an algebra $U$ (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

form
$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

**Problem.** Extract information from $Z$.
**The Dogma.** Use representation theory. In principle finite, but *slow*.

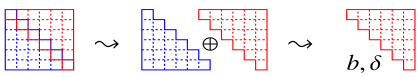
**The (fake) moduli** of Lie algebras on $V$, a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^\epsilon/(\epsilon^{k+1} = 0)$.




**Solvable Approximation.** A quantized universal enveloping algebra (aka "quantum group") is an $\infty$-dimensional inverse limit.

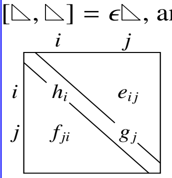
| untrimmed | $\longrightarrow$ | halfway-trimmed | $\longrightarrow$ | almost fully-trimmed |
|---|---|---|---|---|
| $\updownarrow$ | | $\updownarrow$ | | $\updownarrow$ |
| too hard | | "solvable approximation" (a parameter is hidden) | | "finite-type" |

**Recomposing $gl_n$.** Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(◹, b, \delta)$:




$b(◹) = b\colon ◹ \otimes ◹ \to ◹$
$b(◺) \rightsquigarrow \delta\colon ◹ \to ◹ \otimes ◹$

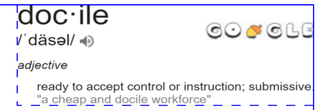
Now define $gl_n^\epsilon := \mathcal{D}(◹, b, \epsilon\delta)$. Schematically, this is $[◹, ◹] = ◹$, $[◺, ◺] = \epsilon◺$, and $[◹, ◺] = ◺ + \epsilon◹$. In detail, it is




$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$ $\quad [f_{ij}, f_{kl}] = \epsilon\delta_{jk} f_{il} - \epsilon\delta_{li} f_{kj}$
$[e_{ij}, f_{kl}] = \delta_{jk}(\epsilon\delta_{j<k} e_{il} + \delta_{il}(h_i + \epsilon g_i)/2 + \delta_{i>l} f_{il})$
$\qquad -\delta_{li}(\epsilon\delta_{k<j} e_{kj} + \delta_{kj}(h_j + \epsilon g_j)/2 + \delta_{k>j} f_{kj})$
$[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik}) e_{jk}$ $\quad [h_i, e_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) e_{jk}$
$[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik}) f_{jk}$ $\quad [h_i, f_{jk}] = \epsilon(\delta_{ij} - \delta_{ik}) f_{jk}$

**Solvable Approximation (2).** At $\epsilon = 1$ and modulo $h = g$, the above is just $gl_n$. By rescaling at $\epsilon \neq 0$, $gl_n^\epsilon$ is independent of $\epsilon$. We let $gl_n^k$ be $gl_n^\epsilon$ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the "$k$-smidgen solvable approximation" of $gl_n$!
Recall that $\mathfrak{g}$ is "solvable" if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2, \mathfrak{g}_2]$, $\ldots$, $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size ◹ algebra.
**Note.** This whole process makes sense for arbitrary semi-simple Lie algebras.

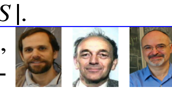
**Definition.** A "docile perturbed Gaussian" in the variables $(z_i)_{i \in S}$ over the ring $R$ is an expression of the form




$$\mathbb{e}^{q^{ij} z_i z_j} P = \mathbb{e}^{q^{ij} z_i z_j} \left( \sum_{k \geq 0} \epsilon^k P_k \right),$$

where all coefficients are in $R$ and where $P$ is a "docile series": $\deg P_k \leq 4k$.
**Docililty Matters!** The rank of the space of docile series to $\epsilon^k$ is polynomial in the number of variables $|S|$.

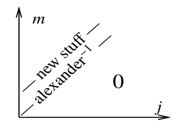
**Theorem** ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of $K$, in the $d$-dimensional representation of $sl_2$. Writing



Melvin, Morton, Garoufalidis

$$\left. \frac{(q^{1/2} - q^{-1/2}) J_d(K)}{q^{d/2} - q^{-d/2}} \right|_{q = e^\hbar} = \sum_{j, m \geq 0} a_{jm}(K) d^j \hbar^m,$$

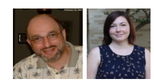
"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $\left( \sum_{m=0}^\infty a_{mm}(K) \hbar^m \right) \cdot \omega(K)(e^\hbar) = 1$.




"Above diagonal" we have **Rozansky's Theorem** [Ro3, (1.2)]:

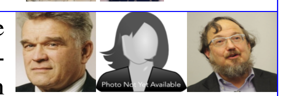
$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1}) \omega(K)(q^d)} \left( 1 + \sum_{k=1}^\infty \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)} \right).$$

**Prior art.** Some amazing computations by Rozansky and Overbay in [Ro2, Ro3] and in [Ov].



Rozansky, Overbay

**Faddeev's Formula** (In as much as we can tell, first appeared w/o proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n - 1}{q - 1}$, with $[n]_q! := [1]_q [2]_q \cdots [n]_q$ and with $\mathbb{e}_q^x := \sum_{n \geq 0} \frac{x^n}{[n]_q!}$, we have




Faddeev Quesne Zagier

$$\log \mathbb{e}_q^x = \sum_{k \geq 1} \frac{(1-q)^k x^k}{k(1-q^k)} = x + \frac{(1-q)^2 x^2}{2(1-q^2)} + \ldots.$$

**Proof.** We have that $\mathbb{e}_q^x = \frac{\mathbb{e}_q^{qx} - \mathbb{e}_q^x}{qx - x}$ ("the $q$-derivative of $\mathbb{e}_q^x$ is itself"), and hence $\mathbb{e}_q^{qx} = (1 + (1-q)x)\mathbb{e}_q^x$, and

$$\log \mathbb{e}_q^{qx} = \log(1 + (1-q)x) + \log \mathbb{e}_q^x.$$

Writing $\log \mathbb{e}_q^x = \sum_{k \geq 1} a_k x^k$ and comparing powers of $x$, we get $q^k a_k = -(1-q)^k/k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. □

***$\mathcal{GDO}$*-Categories.** Given $\mathfrak{g}$ with basis $B = \{x, y, \ldots\}$, consider the following diagram:

$$\begin{tikzcd}
\mathbb{Q} = \hat{\mathcal{U}}_{(q)}\left(\bigoplus_0 \mathfrak{g}\right) \arrow[r, "Z"] & \hat{\mathcal{U}}_{(q)}(\mathfrak{g}) \arrow[r, shift left, "\Delta"] \arrow[d, shift left, "\mathbb{O}(yx\ldots:\,\cdot)"] & \hat{\mathcal{U}}_{(q)}\left(\bigoplus_2 \mathfrak{g}\right) \arrow[l, shift left, "m"] \\
\hat{\mathcal{S}}(\emptyset) \arrow[u] \arrow[r, "Z"] & \hat{\mathcal{S}}(B) \arrow[u, shift left, "\mathbb{O}(xy\ldots:\,\cdot)"] \arrow[r, shift left, "\Delta"] \arrow[loop below, "SW_{xy}"] & \hat{\mathcal{S}}(B_1, B_2) \arrow[l, shift left, "m"] \arrow[u, "\mathbb{O}(y_1x_1\ldots\otimes y_2x_2\ldots:\,\cdot)"']
\end{tikzcd}$$

Hence $Z$, $SW_{xy}$, $m$, $\Delta$, (and likewise $S$ and $\theta$) are morphisms in the *completion* of the monoidal category $\mathcal{F}$ whose objects are finite sets $B$ and whose morphisms are $\mathrm{mor}_{\mathcal{F}}(B, B') := \mathrm{Hom}_{\mathbb{Q}}(\mathcal{S}(B) \to \mathcal{S}(B')) = \mathcal{S}(B^*, B')$ (by convention, $x^* = \xi$, $y^* = \eta$, etc.). Ergo we need to *consolidate* (at least parts of) said completion.

**Aside.** "Consolidate" means "give a finite name to an infinite object, and figure out how to sufficiently manipulate such finite names". E.g., solving $f'' = -f$ we encounter and set $\sum \frac{(-1)^k x^{2k}}{(2k)!} \rightsquigarrow \cos x$, $\sum \frac{(-1)^k x^{2k+1}}{(2k+1)!} \rightsquigarrow \sin x$, and then $\cos^2 x + \sin^2 x = 1$ and $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

**The Composition Law.** If

$$\mathcal{S}(B_0) \xrightarrow[{}^t f \in \mathbb{Q}[\![\zeta_{0i}, z_{1j}]\!]]{f} \mathcal{S}(B_1) \xrightarrow[{}^t g \in \mathbb{Q}[\![\zeta_{1j}, z_{2k}]\!]]{g} \mathcal{S}(B_2)$$

then ${}^t(f /\!/ g) = {}^t(g \circ f) = \left( g|_{\zeta_{1j} \to \partial_{z_{1j}}} f \right)_{z_{1j} = 0}$.

**Examples.**

1. The 1-variable identity map $I\colon \mathcal{S}(z) \to \mathcal{S}(z)$ is given by ${}^t I_1 = \mathbb{e}^{z\zeta}$ and the $n$-variable one by ${}^t I_n = \mathbb{e}^{z_1\zeta_1 + \cdots + z_n\zeta_n}$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Matemale-1804/