

**The Algebras  $H$  and  $H^*$ .** Let  $q = e^{\hbar\epsilon\gamma}$  and set  $H = \langle a, x \rangle / ([a, x] = \gamma x)$  with

$$A = e^{-\hbar\epsilon a}, \quad xA = qAx, \quad S_H(a, A, x) = (-a, A^{-1}, -A^{-1}x),$$

$$\Delta_H(a, A, x) = (a_1 + a_2, A_1A_2, x_1 + A_1x_2)$$

and dual  $H^* = \langle b, y \rangle / ([b, y] = -\epsilon y)$  with

$$B = e^{-\hbar\epsilon yb}, \quad By = qyB, \quad S_{H^*}(b, B, y) = (-b, B^{-1}, -yB^{-1}),$$

$$\Delta_{H^*}(b, B, y) = (b_1 + b_2, B_1B_2, y_1B_2 + y_2).$$

Pairing by  $(a, x)^* = (b, y)$  ( $\Leftrightarrow \langle B, A \rangle = q$ ) making  $\langle y^i b^j, a^j x^k \rangle = \delta_{ij} \delta_{ki} j! [k]_q!$  so  $R = \sum \frac{y^k b^j \otimes a^i x^k}{j! [k]_q!}$ .

**The Algebra  $QU$ .** Using the Drinfel'd double procedure,  $QU_{\gamma, \epsilon} := H^{*cop} \otimes H$  with  $(\phi f)(\psi g) = \langle \psi_1 S^{-1} f_3 \rangle \langle \psi_3, f_1 \rangle \langle \phi \psi_2 \rangle \langle f_2 g \rangle$  and

$$S(y, b, a, x) = (-B^{-1}y, -b, -a, -A^{-1}x),$$

$$\Delta(y, b, a, x) = (y_1 + y_2 B_1, b_1 + b_2, a_1 + a_2, x_1 + A_1 x_2).$$

Note also that  $t := \epsilon a - \gamma b$  is central and can replace  $b$ , and set  $QU = QU_\epsilon = QU_{1, \epsilon}$ .

**The 2D Lie Algebra.** One may show\* that if  $[a, x] = \gamma x$  then  $e^{\epsilon x} e^{a\alpha} = e^{a\alpha} e^{e^{-\gamma\alpha} \epsilon x}$ . Ergo with

$$SW_{ax} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} S(a, x) \begin{array}{c} \xrightarrow{O_{ax}} \\ \xleftarrow{O_{xa}} \end{array} U(a, x)$$

we have  $\widetilde{SW}_{ax} = e^{a\alpha + e^{-\gamma\alpha} \epsilon x}$ .

\* Indeed  $xa = (a - \gamma)x$  thus  $xa^n = (a - \gamma)^n x$  thus  $x e^{a\alpha} = e^{\alpha(a-\gamma)x} = e^{-\gamma\alpha} e^{a\alpha} x$  thus  $x^n e^{a\alpha} = e^{a\alpha} (e^{-\gamma\alpha})^n x^n$  thus  $e^{\epsilon x} e^{a\alpha} = e^{a\alpha} e^{-\gamma\alpha} e^{\epsilon x}$ .

**Faddeev's Formula** (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With  $[n]_q := \frac{q^n - 1}{q - 1}$ , with  $[n]_q! := [1]_q [2]_q \cdots [n]_q$  and with  $e_q^x := \sum_{n \geq 0} \frac{x^n}{[n]_q!}$ , we have

$$\log e_q^x = \sum_{k \geq 1} \frac{(1-q)^k x^k}{k(1-q^k)} = x + \frac{(1-q)^2 x^2}{2(1-q^2)} + \dots$$

**Proof.** We have that  $e_q^x = \frac{e^{qx} - e^x}{qx - x}$  ("the  $q$ -derivative of  $e_q^x$  is itself"), and hence  $e_q^{qx} = (1 + (1-q)x)e_q^x$ , and

$$\log e_q^{qx} = \log(1 + (1-q)x) + \log e_q^x.$$

Writing  $\log e_q^x = \sum_{k \geq 1} a_k x^k$  and comparing powers of  $x$ , we get  $q^k a_k = -(1-q)^k / k + a_k$ , or  $a_k = \frac{(1-q)^k}{k(1-q^k)}$ .  $\square$

**A Full Implementation.**

$\omega\epsilon\beta/\text{Full}$

Utilities

```
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ε_] := ExpandDenominator@ExpandNumerator@Together[
  Expand[ε] /. e^x - e^y -> e^{x+y} /. e^x -> e^{CF[x]}];
Kδ /: Kδ_{i,j} := If[i == j, 1, 0];
E /: E[L1_, Q1_, P1_] ≡ E[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
E /: E[L1_, Q1_, P1_] E[L2_, Q2_, P2_] :=
  E[L1 + L2, Q1 + Q2, P1 + P2];
E[L_, Q_, P_]$_k := E[L, Q, Series[Normal@P, {ε, 0, $k}]]];
```

Zip and Bind

```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z};
(u_{i*})* := (u*)_i;
```

```
collect[sd_SeriesData, ε_] :=
  MapAt[collect[#, ε_] &, sd, 3];
collect[ε_, ε_] := Collect[ε, ε];
Zip_{ } [P_] := P; Zip_{(ε, ε_...)} [P_] :=
  (collect[P // Zip_{(ε, ε_...)}] /. f_ -> ε^{d_} -> ∂_{(ε*, d)} f) /. ε* -> 0
QZip_{ε_List} @E[L_, Q_, P_] :=
  Module[{ε, z, zs, c, ys, ηs, qt, zrule, εrule},
    zs = Table[ε*, {ε, εs}];
    c = CF[Q /. Alternatives@@(εs ∪ zs) -> 0];
    ys = CF@Table[∂_ε(Q /. Alternatives@@zs -> 0), {ε, εs}];
    ηs = CF@Table[∂_z(Q /. Alternatives@@εs -> 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z, ε*} - ∂_{z, ε} Q, {ε, εs}, {z, zs}];
    zrule = Thread[zs -> CF[qt.(zs + ys)]];
    εrule = Thread[εs -> εs + ηs.qt];
    CF /@ E[L, c + ηs.qt.y,
      Det[qt] Zip_{εs} [P /. (zrule ∪ εrule)]];
  ];
U21 = {B_{i-}^{p-} -> e^{-p h y b_i}, B_{i-}^{p-} -> e^{-p h y b}, T_{i-}^{p-} -> e^{p h t_i},
  T_{i-}^{p-} -> e^{p h t}, A_{i-}^{p-} -> e^{p y a_i}, A_{i-}^{p-} -> e^{p y a}};
L2U = {e^{c_{-} b_i + d_{-}} -> B_{i-}^{-c/(h y)} e^d, e^{c_{-} b + d_{-}} -> B^{-c/(h y)} e^d,
  e^{c_{-} t_i + d_{-}} -> T_{i-}^{c/h} e^d, e^{c_{-} t + d_{-}} -> T^{c/h} e^d,
  e^{c_{-} a_i + d_{-}} -> A_{i-}^{c/y} e^d, e^{c_{-} a + d_{-}} -> A^{c/y} e^d,
  e^{ε_{-}} -> e^{Expand@ε}};
LZip_{ε_List} @E[L_, Q_, P_] :=
  Module[{ε, z, zs, c, ys, ηs, lt, zrule, L1, L2, Q1, Q2},
    zs = Table[ε*, {ε, εs}];
    c = L /. Alternatives@@(εs ∪ zs) -> 0;
    ys = Table[∂_ε(L /. Alternatives@@zs -> 0), {ε, εs}];
    ηs = Table[∂_z(L /. Alternatives@@εs -> 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z, ε*} - ∂_{z, ε} L, {ε, εs}, {z, zs}];
    zrule = Thread[zs -> lt.(zs + ys)];
    L2 = (L1 = c + ηs.zs /. zrule) /. Alternatives@@zs -> 0;
    Q2 = (Q1 = Q /. U21 /. zrule) /. Alternatives@@zs -> 0;
    CF /@ E[L2, Q2, Det[lt] e^{-L2-Q2}
      Zip_{εs} [e^{L1+Q1} (P /. U21 /. zrule)]] // L2U];
B_{ } [L_, R_] := LR;
B_{is_} [L_{E}, R_{E}] := Module[{n}, Times[
  L /. Table[(v : b | B | t | T | a | x | y)_i -> v_{nei}, {i, {is}}],
  R /. Table[(v : β | τ | α | A | ξ | η)_i -> v_{nei}, {i, {is}}]
] // LZipJoin@Table[{β_{nei}, τ_{nei}, a_{nei}}, {i, {is}}] //
  QZipJoin@Table[{ε_{nei}, y_{nei}}, {i, {is}}];
B_{is_} [L_, R_] := B_{is} [L, R];
```

"Define" code

```
SetAttributes[Define, HoldAll];
Define[def_, defs_] := (Define[def]; Define[defs]);
```