## Demo Programs for 0-Co. ωεβ/Demo

**The $R$-matrices**

$$\mathtt{R}^{+}_{0,i\_,j\_} := \mathbb{E}\left[b_i\, c_j + b_i^{-1}\left(e^{b_i}-1\right) u_i\, w_j\right];$$
$$\mathtt{R}^{-}_{0,i\_,j\_} := \mathbb{E}\left[-b_i\, c_j + b_i^{-1}\left(e^{-b_i}-1\right) u_i\, w_j\right];$$

**Utilities**

$$\mathtt{CF}[\omega\_ .\ \mathbb{E}[Q\_]] := \mathtt{Simplify}[\omega]\ \mathbb{E}[\mathtt{Simplify}[Q]];$$
$$\mathbb{E}\ /: \mathbb{E}[Q1\_]\ \mathbb{E}[Q2\_] := \mathtt{CF@}\mathbb{E}[Q1+Q2];$$
$$\omega 1\_ .\ \mathbb{E}[Q1\_] \equiv \omega 2\_ .\ \mathbb{E}[Q2\_] := \mathtt{Simplify}[\omega 1 == \omega 2 \wedge Q1 == Q2];$$

**Normal Ordering Operators**

$$\mathtt{N}_{(x:w|u)_{i\_}\, c_{j\_} \to k\_}[\omega\_ .\ \mathbb{E}[Q\_]] := \mathtt{CF}\Big[\omega\, \mathbb{E}[e^{\gamma}\, \alpha\, x_k + \gamma\, c_k + (Q\ /.\ c_j \mid x_i \to 0)]\ /.\ \{\gamma \to \partial_{c_j} Q,\ \alpha \to \partial_{x_i} Q\}\Big];$$
$$\mathtt{N}_{w_{i\_}\, u_{j\_} \to k\_}[\omega\_ .\ \mathbb{E}[Q\_]] := \mathtt{CF}\Big[\nu\, \omega\, \mathbb{E}[-b_k\, \nu\, \alpha\, \beta + \nu\, \beta\, u_k + \nu\, \alpha\, w_k + \nu\, \delta\, u_k\, w_k + (Q\ /.\ w_i \mid u_j \to 0)]\ /.\ \nu \to (1+b_k\, \delta)^{-1}\ /.\ \{\alpha \to \partial_{w_i} Q\ /.\ u_j \to 0,\ \beta \to \partial_{u_j} Q\ /.\ w_i \to 0,\ \delta \to \partial_{w_i, u_j} Q\}\Big];$$

**Stitching**

$$\mathtt{m}_{i\_,j\_\to k\_}[Z\_] := \mathtt{Module}\Big[\{x, z\},\ \mathtt{CF}\Big[\big(Z\ //\ \mathtt{N}_{w_i\, u_j \to x}\ //\ \mathtt{N}_{c_i\, u_x \to x}\ //\ \mathtt{N}_{w_x\, c_j \to x}\big)\ /.\ z\_{i|j|x} \to z_k\Big]\Big]$$

**Some calculations for $T_0$**

$$\mathtt{T0} = \mathtt{R}^{+}_{0,5,1}\, \mathtt{R}^{+}_{0,2,4}\, \mathtt{R}^{-}_{0,3,6}$$

$$\mathbb{E}\left[b_5\, c_1 + b_2\, c_4 - b_3\, c_6 + \frac{(-1+e^{b_5})\, u_5\, w_1}{b_5} + \frac{(-1+e^{b_2})\, u_2\, w_4}{b_2} + \frac{(-1+e^{-b_3})\, u_3\, w_6}{b_3}\right]$$

$$\mathtt{T0}\ //\ \mathtt{m}_{1,2\to1}\ //\ \mathtt{m}_{3,4\to3}\ //\ \mathtt{m}_{3,5\to3}\ //\ \mathtt{m}_{3,6\to3}$$

$$\frac{1}{1-(-1+e^{b_1})\,(-1+e^{b_3})}\ \mathbb{E}\Big[b_3\, c_1 + b_1\, c_3 - b_3\, c_3 + \frac{e^{b_3}\,(-1+e^{b_1})\,(-1+e^{b_3})\, u_1\, w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\, b_1} - \frac{e^{b_1}\,(-1+e^{b_3})\, u_3\, w_1}{(-1+(-1+e^{b_1})\,(-1+e^{b_3}))\, b_3} - \frac{e^{-b_3}\,(-1+e^{b_3})\, u_3\, w_3}{b_3} - \frac{e^{-b_3}\,(-1+e^{b_1})\,(-e^{b_3}\, b_3\, u_1 + e^{b_1}\,(-1+e^{b_3})\, b_1\, u_3)\, w_3}{b_1\,(b_3-(-1+e^{b_1})\,(-1+e^{b_3})\, b_3)}\Big]$$

**Verifying meta-associativity**

$$\mathtt{Q0} = \mathbb{E}[\mathtt{Sum}[f_i\, c_i, \{i, 3\}] + \mathtt{Sum}[f_{i,j}\, u_i\, w_j, \{i, 3\}, \{j, 3\}]]$$

$$\mathbb{E}[c_1\, f_1 + c_2\, f_2 + c_3\, f_3 + u_1\, w_1\, f_{1,1} + u_1\, w_2\, f_{1,2} + u_1\, w_3\, f_{1,3} + u_2\, w_1\, f_{2,1} + u_2\, w_2\, f_{2,2} + u_2\, w_3\, f_{2,3} + u_3\, w_1\, f_{3,1} + u_3\, w_2\, f_{3,2} + u_3\, w_3\, f_{3,3}]$$

$$(\mathtt{Q0}\ //\ \mathtt{m}_{1,2\to1}\ //\ \mathtt{m}_{1,3\to1}) \equiv (\mathtt{Q0}\ //\ \mathtt{m}_{2,3\to2}\ //\ \mathtt{m}_{1,2\to1})$$

```
True
```

**Testing R3**

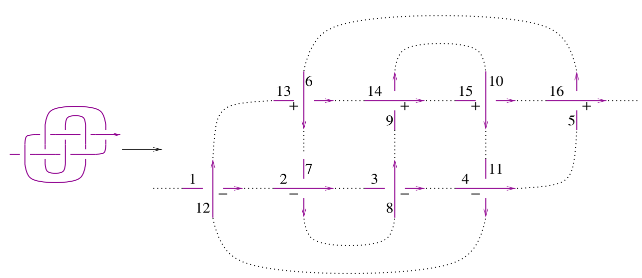
$$\mathtt{t1} = \mathtt{R}^{+}_{0,1,2}\, \mathtt{R}^{+}_{0,3,4}\, \mathtt{R}^{+}_{0,5,6}\ //\ \mathtt{m}_{3,5\to x}\ //\ \mathtt{m}_{1,6\to y}\ //\ \mathtt{m}_{2,4\to z}$$

$$\mathbb{E}\left[b_x\, c_y + b_x\, c_z + b_y\, c_z + \frac{e^{b_x}\,(-1+e^{b_y})\, u_y\, w_z}{b_y} + \frac{(-1+e^{b_x})\, u_x\,(w_y+w_z)}{b_x}\right]$$

$$\mathtt{t1} \equiv (\mathtt{R}^{+}_{0,1,2}\, \mathtt{R}^{+}_{0,3,4}\, \mathtt{R}^{+}_{0,5,6}\ //\ \mathtt{m}_{1,3\to x}\ //\ \mathtt{m}_{2,5\to y}\ //\ \mathtt{m}_{4,6\to z})$$

```
True
```

$8_{17}$




$$\mathtt{z1} = \mathtt{R}^{-}_{0,12,1}\, \mathtt{R}^{-}_{0,2,7}\, \mathtt{R}^{-}_{0,8,3}\, \mathtt{R}^{-}_{0,4,11}\, \mathtt{R}^{+}_{0,16,5}\, \mathtt{R}^{+}_{0,6,13}\, \mathtt{R}^{+}_{0,14,9}\, \mathtt{R}^{+}_{0,10,15};$$
$$\mathtt{Do}[\mathtt{z1} = (\mathtt{z1}\ //\ \mathtt{m}_{1,n\to1})\ /.\ b\_ \to b,\ \{n, 2, 16\}];$$
$$\{\mathtt{CF@z1},\ \mathtt{KnotData}[\{8, 17\}, \text{"AlexanderPolynomial"}][t]\}$$

$$\left\{-\frac{e^{3b}\, \mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11-\frac{1}{t^3}+\frac{4}{t^2}-\frac{8}{t}-8t+4t^2-t^3\right\}$$

## Demo Programs for 1-Co. ωεβ/Demo

**The $R$-matrices**

$$\mathtt{R}^{+}_{i\_,j\_} := \mathbb{E}\left[1,\ \mathtt{Log}[t_i]\, c_j,\ v_i\, w_j,\ v_i\, c_i\, w_j + c_i\, c_j + v_i^2\, w_j^2/4\right];$$
$$\mathtt{R}^{-}_{i\_,j\_} := \mathbb{E}\left[1,\ -\mathtt{Log}[t_i]\, c_j,\ -t_i^{-1}\, v_i\, w_j,\ t_i^{-1}\, v_i\, c_j\, w_j - c_i\, c_j - t_i^{-2}\, v_i^2\, w_j^2/4\right];$$
$$\left(\mathtt{ur}_{i\_} := \mathbb{E}\left[t_i^{-1/2}, 0, 0, c_i\, t_i^{-2}\right];\ \mathtt{nr}_{i\_} := \mathbb{E}\left[t_i^{1/2}, 0, 0, -c_i\, t_i^{2}\right];\right)$$

**The Generators**

**Differential Polynomials**

$$\mathtt{DP}_{x\_\to D_{\alpha\_},\, y\_\to D_{\beta\_}}[P\_][f\_] := \ (\ast\ \text{means } P[\partial_\alpha, \partial_\beta][f]\ \ast)$$
$$\mathtt{Total}[\mathtt{CoefficientRules}[P, \{x, y\}]\ /.\ (\{m\_, n\_\} \to c\_) \mapsto c\, \mathtt{D}[f, \{\alpha, m\}, \{\beta, n\}]]$$

**Utilities**

$$\mathtt{CF}[\mathcal{E}\_\mathbb{E}] := \mathtt{Expand}\ /@\ \mathtt{Together}\ /@\ \mathcal{E};$$
$$\mathbb{E}\ /: \mathbb{E}[\omega 1\_, L1\_, Q1\_, P1\_]\ \mathbb{E}[\omega 2\_, L2\_, Q2\_, P2\_] := \mathtt{CF@}\mathbb{E}\left[\omega 1\, \omega 2,\ L1+L2,\ \omega 2\, Q1 + \omega 1\, Q2,\ \omega 2^4\, P1 + \omega 1^4\, P2\right];$$

**Normal Ordering Operators**

$$\mathtt{N}_{c_{j\_}\ (x:v|w)_{i\_} \to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \mathtt{With}\Big[\{q = e^{\gamma}\, \beta\, x_k + \gamma\, c_k\},\ \mathtt{CF}\Big[\mathbb{E}\Big[\omega,\ \gamma\, c_k + (L\ /.\ c_j \to 0),\ \omega\, e^{\gamma}\, \beta\, x_k + (Q\ /.\ x_i \to 0),\ e^{-q}\, \mathtt{DP}_{c_j\to D_\gamma,\, x_i\to D_\beta}[P][e^{q}]\Big]\ /.\ \{\gamma \to \partial_{c_j} L,\ \beta \to \omega^{-1}\, \partial_{x_i} Q\}\Big]\Big];$$

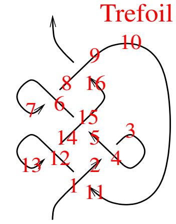
$$\mathtt{N}_{w_{i\_}\ v_{j\_} \to k\_}[\mathbb{E}[\omega\_, L\_, Q\_, P\_]] := \mathtt{With}\Big[\{q = ((1-t_k)\, \alpha\, \beta + \beta\, v_k + \alpha\, w_k + \delta\, v_k\, w_k)/\mu\},\ \mathtt{CF}\Big[\mathbb{E}\Big[\mu\, \omega,\ L,\ \mu\, \omega\, q + \mu\, (Q\ /.\ w_i \mid v_j \to 0),\ \mu^4\, e^{-q}\, \mathtt{DP}_{w_i\to D_\alpha,\, v_j\to D_\beta}[P][e^{q}] + \omega^4\, \Lambda[k]\Big]\ /.\ \mu \to 1 + (t_k - 1)\, \delta\ /.\ \{\alpha \to \omega^{-1}\,(\partial_{w_i} Q\ /.\ v_j \to 0),\ \beta \to \omega^{-1}\,(\partial_{v_j} Q\ /.\ w_i \to 0),\ \delta \to \omega^{-1}\, \partial_{w_i, v_j} Q\}\Big]\Big];$$

**Stitching**

$$\mathtt{m}_{i\_,j\_\to k\_}[Z\_\mathbb{E}] := \mathtt{Module}\Big[\{x, z\},\ \mathtt{CF}\Big[\big(Z\ //\ \mathtt{N}_{w_i\, v_j \to x}\ //\ \mathtt{N}_{c_i\, v_x \to x}\ //\ \mathtt{N}_{w_x\, c_j \to x}\big)\ /.\ z\_{i|j|x} \to z_k\Big]\Big]$$

**The 0-Framed Trefoil**

$$\mathtt{z2} = \mathtt{R}^{+}_{1,11}\, \mathtt{R}^{-}_{4,2}\, \mathtt{nr}_3\, \mathtt{R}^{+}_{15,5}\, \mathtt{R}^{-}_{6,8}\, \mathtt{ur}_7\, \mathtt{R}^{+}_{9,16}\, \mathtt{nr}_{10}\, \mathtt{R}^{-}_{12,14}\, \mathtt{ur}_{13};$$
$$(\mathtt{Do}[\mathtt{z2} = \mathtt{z2}\ //\ \mathtt{m}_{1,k\to1},\ \{k, 2, 16\}];$$
$$\mathtt{z2} = \mathtt{z2}\ /.\ a\_1 \mapsto a)$$




$$\mathbb{E}\Big[-1+\frac{1}{t}+t,\ 0,\ 0,\ 16+\frac{2c}{t^4}-\frac{1}{t^3}-\frac{6c}{t^3}+\frac{4}{t^2}+\frac{10c}{t^2}-\frac{10}{t}-\frac{8c}{t}-18t+8ct+14t^2-10ct^2-7t^3+6ct^3+2t^4-2ct^4+2vw-\frac{2vw}{t^4}+\frac{4vw}{t^3}-\frac{6vw}{t^2}+\frac{2vw}{t}-6tvw+4t^2vw-2t^3vw\Big]$$

**The Λόγος**

$$\Lambda[k\_] := \Big((t_k-1)\,\big(2\,(\alpha\,\beta+\delta\,\mu)^2-\alpha^2\,\beta^2\big) - 4\, v_k\, c_k\, w_k\, \delta^2\, \mu^2 - \delta\,(1+\mu)\,\big(w_k^2\,\alpha^2 + v_k^2\,\beta^2\big) - v_k^2\, w_k^2\, \delta^3\,(1+3\mu) - 2\,\big(\alpha\,\beta + 2\,\delta\,\mu + v_k\, w_k\, \delta^2\,(1+2\mu) + 2\, c_k\, \delta\, \mu^2\big)\,(w_k\,\alpha + v_k\,\beta) - 4\,\big(c_k\,\mu^2 + v_k\, w_k\, \delta\,(1+\mu)\big)\,(\alpha\,\beta+\delta\,\mu)\Big)\,(1+t_k)/4;$$

**Questions and To Do List.** • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on $R$. • Is there a canonical $R$? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" $q$-algebra. • $k$-smidgen $sl_n$, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.

This is http://www.math.toronto.edu/~drorbn/Talks/MIT-1612/. Better videos at .../Indiana-1611/, .../LesDiablerets-1608/