

$\mathcal{D}: \text{Hom}(U^{\otimes \Sigma} \rightarrow U^{\otimes S}) \rightarrow \mathbb{Q}[\eta_\Sigma, \beta_\Sigma, \alpha_\Sigma, \xi_\Sigma, \gamma_S, b_S, a_S, x_S]$. The PBW theorem for CU (always in the $ybax$ order), or its quantum analog for QU , say that if $U = CU$ or QU then $U^{\otimes S}$ is isomorphic as a vector space to $\mathbb{Q}[y_i, b_i, a_i, x_i]_{i \in S}[[\hbar]]$; so it is enough to understand $\text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B])$ for finite sets A and B .

Claim. $F \in \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B]) \xrightarrow{\cong} \mathbb{Q}[z_B][[\zeta_A]] \ni \mathcal{F}$ via

$$\mathcal{D}(F) := \sum_{n \in \mathbb{N}^A} \frac{\zeta_A^n}{n!} F(z_A^n) = F\left(\bigoplus_{\sigma \in \mathcal{A}} \zeta_{\sigma}^{\sigma}\right) = \mathcal{F},$$

$$\mathcal{D}^{-1}(\mathcal{F})(p) = \left(p|_{z_a \rightarrow \partial_{z_a} \mathcal{F}}\right)_{\zeta_a=0} \quad \text{for } p \in \mathbb{Q}[z_A].$$

Claim. Assuming convergence, if $F \in \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B])$, $G \in \text{Hom}(\mathbb{Q}[z_B] \rightarrow \mathbb{Q}[z_C])$, $\mathcal{F} = \mathcal{D}(F)$, and $\mathcal{G} = \mathcal{D}(G)$, then

$$\mathcal{D}(F \circ G) = \left(\mathcal{F}|_{z_i \rightarrow \partial_{z_i} \mathcal{G}}\right)_{\zeta_i=0}.$$

And so the title of the talk finally makes sense!

Example. $\mathcal{D}(id: U \rightarrow U) = e^{\eta y + \beta b + \alpha a + \xi x}$.

Example. Let $c\Delta_{jk}^i: CU^{\otimes(i)} \rightarrow CU^{\otimes(j,k)}$ be the standard coproduct, given by $c\Delta_{jk}^i(y_i, b_i, a_i, x_i) = (y_j + y_k, b_j + b_k, a_j + a_k, x_j + x_k)$. Then

$$\begin{aligned} \mathcal{D}(c\Delta_{jk}^i) &= c\Delta_{jk}^i(e^{\eta_i y_i + \beta_i b_i + \alpha_i a_i + \xi_i x_i}) \\ &= e^{\eta_i(y_j + y_k) + \beta_i(b_j + b_k) + \alpha_i(a_j + a_k) + \xi_i(x_j + x_k)}. \end{aligned}$$

Example. The standard commutative product m_k^{ij} of polynomials is given by $z_i, z_j \rightarrow z_k$. Hence $\mathcal{D}(m_k^{ij}) =$

$$m_k^{ij}(e^{\zeta_i z_i + \zeta_j z_j}) = e^{(\zeta_i + \zeta_j) z_k} \quad \begin{array}{ccc} \mathbb{Q}[z]_i \otimes \mathbb{Q}[z]_j & \xrightarrow{m_k^{ij}} & \mathbb{Q}[z]_k \\ \parallel & & \parallel \\ \mathbb{Q}[z_i, z_j] & \xrightarrow{m_k^{ij}} & \mathbb{Q}[z_k] \end{array}$$

A real DoPeGDO Example. Let $cm_k^{ij}: CU_i \otimes CU_j \rightarrow CU_k$ be ‘‘classical multiplication’’ for sl_{2+}^ϵ , and let $\mathbb{O}_i: \mathbb{Q}[y_i, b_i, a_i, x_i] \rightarrow CU_i$ be the PBW ordering map.

$$\begin{array}{ccc} CU_i \otimes CU_j & \xrightarrow{cm_k^{ij}} & CU_k \\ \uparrow \mathbb{O}_{i,j} & & \uparrow \mathbb{O}_k \\ \mathbb{Q}[y_i, b_i, a_i, x_i, y_j, b_j, a_j, x_j] & & \mathbb{Q}[y_k, b_k, a_k, x_k] \end{array}$$

Claim. Let (all brawn and no brains)

$$\begin{aligned} \Lambda &= \left(\eta_i + \frac{e^{-\alpha_i - \epsilon \beta_i} \eta_j}{1 + \epsilon \eta_j \xi_i}\right) y_k + \left(\beta_i + \beta_j + \frac{\log(1 + \epsilon \eta_j \xi_i)}{\epsilon}\right) b_k + \\ &\quad (\alpha_i + \alpha_j + \log(1 + \epsilon \eta_j \xi_i)) a_k + \left(\frac{e^{-\alpha_j - \epsilon \beta_j} \xi_i}{1 + \epsilon \eta_j \xi_i} + \xi_j\right) x_k \end{aligned}$$

Then $e^{\eta_i y_i + \beta_i b_i + \alpha_i a_i + \xi_i x_i + \eta_j y_j + \beta_j b_j + \alpha_j a_j + \xi_j x_j} \circ \mathbb{O}_{i,j} \circ cm_k^{ij} = e^\Lambda \circ \mathbb{O}_k$, and hence $\mathcal{D}(cm_k^{ij}) = e^\Lambda$ and cm_k^{ij} is DoPeGDO.

Proof. We compute in a faithful 2D representation $z \mapsto \hat{z}$ of CU : (ωεβ/cm)

HL [\mathcal{E}_-] := `Style[\mathcal{E} , Background → If[TrueQ@ \mathcal{E} , #, #]];`
 $\{\hat{y} = \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}, \hat{b} = \begin{pmatrix} 0 & 0 \\ 0 & -\epsilon \end{pmatrix}, \hat{a} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \hat{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\};$
 $\text{HL} / @ \{\hat{a} \cdot \hat{x} - \hat{x} \cdot \hat{a} = \hat{x}, \hat{a} \cdot \hat{y} - \hat{y} \cdot \hat{a} = -\hat{y}, \hat{b} \cdot \hat{y} - \hat{y} \cdot \hat{b} = -\epsilon \hat{y}, \hat{b} \cdot \hat{x} - \hat{x} \cdot \hat{b} = \epsilon \hat{x}, \hat{x} \cdot \hat{y} - \hat{y} \cdot \hat{x} = \hat{b} + \epsilon \hat{a}\}$

{True, True, True, True, True}

HL@Simplify@With[{ \mathbb{E} = MatrixExp},

$\mathbb{E}[\eta_i \hat{y}] \cdot \mathbb{E}[\beta_i \hat{b}] \cdot \mathbb{E}[\alpha_i \hat{a}] \cdot \mathbb{E}[\xi_i \hat{x}] \cdot \mathbb{E}[\eta_j \hat{y}] \cdot \mathbb{E}[\beta_j \hat{b}] \cdot \mathbb{E}[\alpha_j \hat{a}] \cdot \mathbb{E}[\xi_j \hat{x}] = \mathbb{E}[\hat{y} \partial_{y_k} \Lambda] \cdot \mathbb{E}[\hat{b} \partial_{b_k} \Lambda] \cdot \mathbb{E}[\hat{a} \partial_{a_k} \Lambda] \cdot \mathbb{E}[\hat{x} \partial_{x_k} \Lambda]$]

True

Series [$\Lambda, \{\epsilon, 0, 1\}$]

$$\begin{aligned} &(\mathbf{a}_k (\alpha_i + \alpha_j) + \mathbf{y}_k (\eta_i + e^{-\alpha_i} \eta_j) + \\ &\quad \mathbf{b}_k (\beta_i + \beta_j + \eta_j \xi_i) + \mathbf{x}_k (e^{-\alpha_j} \xi_i + \xi_j)) + \\ &\quad \left(\mathbf{a}_k \eta_j \xi_i - \frac{1}{2} \mathbf{b}_k \eta_j^2 \xi_i^2 - e^{-\alpha_i} \mathbf{y}_k \eta_j (\beta_i + \eta_j \xi_i) - \right. \\ &\quad \left. e^{-\alpha_j} \mathbf{x}_k \xi_i (\beta_j + \eta_j \xi_i)\right) \epsilon + 0[\epsilon]^2 \end{aligned}$$

(Shame, but this technique fails for QU).

Claim. In QU , R is DoPeGDO.

Proof. Recall that with $q = e^{\hbar \epsilon}$,

$$R = \sum \hbar^{j+k} y^k b^j \otimes a^j x^k / j! k! q! = \mathbb{O}\left(e^{\hbar b_1 a_2} e^{\hbar y_1 x_2}\right).$$

Now expand $e^{\hbar y_1 x_2}$ in powers of ϵ using:

Faddeev’s Formula (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n - 1}{q - 1}$, with $[n]_q! := [1]_q [2]_q \cdots [n]_q$ and with $e_q^x := \sum_{n \geq 0} \frac{x^n}{[n]_q!}$, we have

$$\log e_q^x = \sum_{k \geq 1} \frac{(1 - q)^k x^k}{k(1 - q^k)} = x + \frac{(1 - q)^2 x^2}{2(1 - q^2)} + \dots$$

Proof. We have that $e_q^x = \frac{e^{qx} - e^x}{qx - x}$ (‘‘the q -derivative of e_q^x is itself’’), and hence $e_q^{qx} = (1 + (1 - q)x)e_q^x$, and

$$\log e_q^{qx} = \log(1 + (1 - q)x) + \log e_q^x.$$

Writing $\log e_q^x = \sum_{k \geq 1} a_k x^k$ and comparing powers of x , we get $q^k a_k = -(1 - q)^k / k + a_k$, or $a_k = \frac{(1 - q)^k}{k(1 - q^k)}$. \square

Compositions (2). Recall that with all indices i running in some set B ,

$$\mathcal{F} \circ \mathcal{G} = \left(\mathcal{F}|_{z_i \rightarrow \partial_{z_i} \mathcal{G}}\right)_{\zeta_i=0} \stackrel{(1)}{=} e^{\sum \partial_{z_i} \partial_{z_i} (\mathcal{F} \mathcal{G})} \Big|_{z_i = \zeta_i = 0}, \quad \begin{array}{l} (1) \text{ Strictly speaking,} \\ \text{true only when} \\ B \cap (A \cup C) = \emptyset. \end{array}$$

so in general we wish to understand

$$[F: \mathcal{E}]_B := e^{\frac{1}{2} \sum_{i,j \in B} F_{ij} \partial_{z_i} \partial_{z_j} \mathcal{E}} \quad \text{and} \quad \langle F: \mathcal{E} \rangle_B := [F: \mathcal{E}]_B|_{z_B \rightarrow 0},$$

where \mathcal{E} is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where \mathcal{E} has no B - B quadratic part:

Lemma 1. With convergences left to the reader,

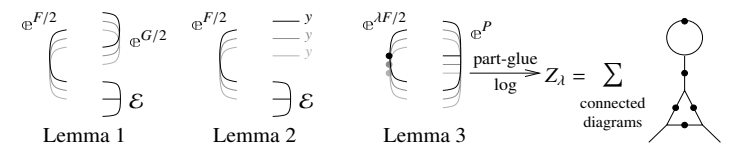
$$\left\langle F: \mathcal{E} e^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1}: \mathcal{E} \right\rangle_B.$$

The next lemma dispatches the case where \mathcal{E} has a B -linear part:

Lemma 2. $\left\langle F: \mathcal{E} e^{\sum_{i \in B} y_i z_i} \right\rangle_B = e^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F: \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$. Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F: e^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$



Complexity to ϵ^k , for an n -xing width w knot (by [LT], $w \in O(\sqrt{n})$), is $O(n^2 w^{2k+2} \log n) = O(n^{k+3} \log n)$ integer operations.