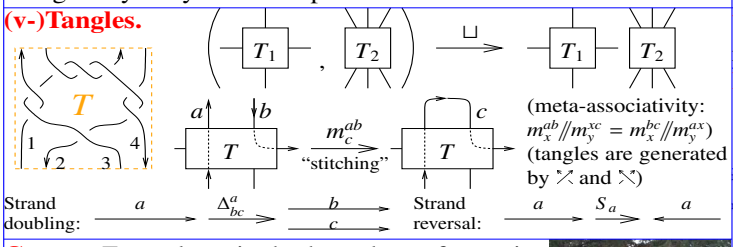
Dror Bar-Natan: Talks: Macquarie-191016: ωεβ:=http://drorbn.net/mac19/

Algebraic Knot Theory

Abstract. This will be a very "light" talk: I will explain why about 13 years ago, in order to have a say on some problems in knot theory, I've set out to find tangle invariants with some nice compositional properties. In other talks in Sydney (ωεβ/talks) I have explained / will explain how such invariants were found - though they are yet to be explored and utilized.

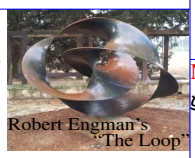
(v-)Tangles.

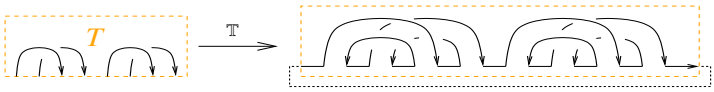


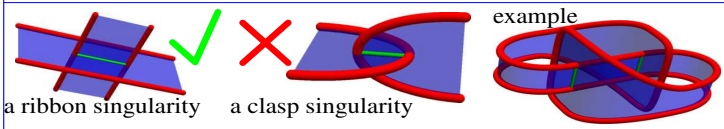
Genus.

Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via





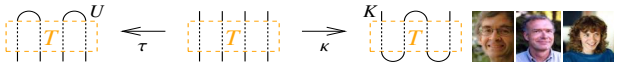


A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

Theorem. K is ribbon iff it is κT for a tangle T for which τT is the untangle U.

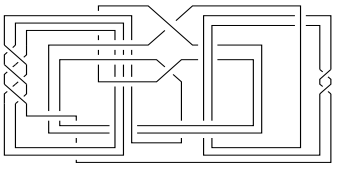


Gompf, Scharlemann, Thompson [GST]

$U \in \mathcal{T}_n$ $\quad$ $1 \in \mathcal{A}_n$

$\mathcal{T}_{2n} \xrightarrow{z} \mathcal{A}_{2n}$ with $\mathcal{R} := \kappa(\tau^{-1}(1))$

ribbon $K \in \mathcal{T}_1$ $\quad$ $z(K) \in \mathcal{R} \subseteq \mathcal{A}_1$



Faster is better, leaner is meaner!

The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c} \omega & S \\ \hline S & A \end{array}\right\}$ with $R_S := \mathbb{Z}(\{T_a : a \in S\})$:

$\left(a \overset{\nearrow}{\searrow} b,\ b \overset{\nearrow}{\searrow} a\right) \to \begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-T_a^{\pm 1} \\ b & 0 & T_a^{\pm 1} \end{array}$ $\quad$ $T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2 \\ \hline S_1 & A_1 & 0 \\ S_2 & 0 & A_2 \end{array}$

$\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \xrightarrow[T_a, T_b \to T_c]{m_c^{ab}} \left\{\begin{array}{c|cc} (1-\beta)\omega & c & S \\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta} \\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta} \end{array}\right.$

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

Strand Doubling and Reversal.

$\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array} \xrightarrow[\substack{\mu:=T_a-1 \\ \nu:=\alpha-\sigma_a T_a \\ T_a\to T_b T_c}]{q\Delta_{bc}^a} \left\{\begin{array}{c|ccc} \omega & b & c & S \\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b-1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu \\ c & (T_c-1)\nu/\mu & (\alpha-\sigma_a T_a-\nu T_c)/\mu & (T_c-1)\theta/\mu \\ S & \phi & \phi & \Xi \end{array}\right.$

$\downarrow dS^a \; T_a\to T_a^{-1}$

$\left\{\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi-\phi\theta)/\alpha \end{array}\right.$

Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left(a \overset{\nearrow}{\searrow} b,\ b \overset{\nearrow}{\searrow} a\right) = (a \to 1,\ b \to t_a^{\pm 1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma/\!\!/m_c^{ab} = (\sigma \setminus \{a,b\}) \cup (c \to \sigma_a\sigma_b)|_{t_a,t_b\to t_c}$.

Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Implementation key idea:

$(\omega, A = (\alpha_{ab})) \leftrightarrow$
$(\omega, \lambda = \sum \alpha_{ab} t_a h_b)$

ωεβ/AlexDemo

```
rCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
  Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
  S = Union@Cases[Γ[ω, λ], (h | t)_a_ :> a, ∞];
  M = Outer[Factor[∂_{h_a} ∂_{t_a} λ] &, S, S];
  M = Prepend[M, t_a & /@ S] // Transpose;
  M = Prepend[M, Prepend[h_a & /@ S, ω]];
  M // MatrixForm];
```

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Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 ⋆ ω2, λ1 + λ2];
m_{a_,b_}→c_[Γ[ω_, λ_]] := Module[{α, β, γ, δ, ϵ, ϕ, ψ, Ξ, μ},
  {{α, β, ϵ}, {γ, δ, ϵ}, {ϕ, ψ, Ξ}} = {{∂_{t_a} λ, ∂_{t_a} λ, ∂_{t_a} λ},
    {∂_{t_b} λ, ∂_{t_b} λ, ∂_{t_b} λ}, {∂_{h_a} λ, ∂_{h_b} λ, λ}} /. {(t | h)_a|b → 0};
  Γ[(μ = 1 - β) ω, {t_c, 1}.{{γ + α δ/μ, ϵ + δ θ/μ}, {ϕ + α ψ/μ, Ξ + ψ θ/μ}}.{h_c, 1}]
   /. {T_a → T_c, T_b → T_c} // rCollect];
Rp_{a_,b_} := Γ[1, {t_a, t_b}.{{1, 1 - T_a}, {0, T_a}}.{h_a, h_b}];
Rm_{a_,b_} := Rp_{a,b} /. T_a → 1/T_a;
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Meta-Associativity

```
ξ = Γ[ω, {t1, t2, t3, ts}.{{α11, α12, α13, θ1}, {α21, α22, α23, θ2},
    {α31, α32, α33, θ3}, {ϕ1, ϕ2, ϕ3, Ξ}}.{h1, h2, h3, hs}];
(ξ // m12→1 // m13→1) == (ξ // m23→2 // m12→1)
```

True

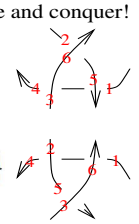
Runs.

R3

. . . divide and conquer!

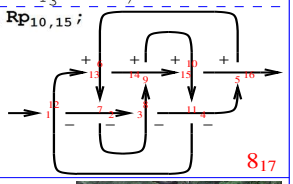
```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}\right\}$$



```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

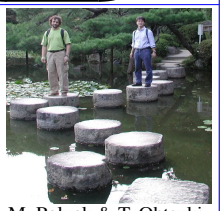
$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8\,T_1 + 4\,T_1^2 - T_1^3 & h_1 \\ t_1 & 1 \end{pmatrix}$$



Fact. Γ is better viewed as an invariant of a certain class of 2D knotted objects in $\mathbb{R}^4$ [BND, BN].

Fact. Γ is the "0-loop" part of an invariant that generalizes to "n-loops" (1D tangles only, see further talks and future publications with van der Veen).

Speculation. Stepping stones to categorification?



M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

Ask me about geography vs. identity!

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant*, ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: w-Knots and the Alexander Polynomial*, Alg. and Geom. Top. **16-2** (2016) 1063–1133, arXiv:1405.1956, ωεβ/WKO1.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial*, J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures*, Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[Vo] H. Vo, *Alexander Invariants of Tangles via Expansions*, University of Toronto Ph.D. thesis, ωεβ/Vo.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Macquarie-191016/