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My talk yesterday:

More Dror: [ωεβ/talks](http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/)

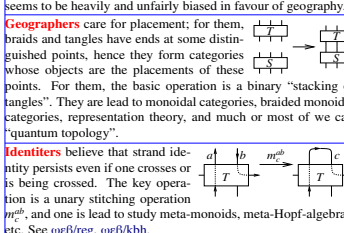
Dror Bar-Natan: Talks: Toronto-1912 [ωεβ:=http://drorbn.net/to19/](http://drorbn.net/to19/)

**Geography vs. Identity**  
Thanks for inviting me to the *Topology* session!

**Abstract.** Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

**Geographers** care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary “stacking of tangles”. They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call “quantum topology”.

**Identifiers** believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation  $m_c^{ab}$ , and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See [ωεβ/leg](http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/), [ωεβ/kbh](http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/).

**Braids.**  
  
Geography:  $GB := \langle \gamma_i \mid \gamma_i \gamma_k = \gamma_k \gamma_i \text{ when } |i-k| > 1 \rangle = B$ .  
(better topology!)  
(captures quantum algebra!)

**Identity:**  
 $IB := \langle \sigma_{ij} \mid \sigma_{ij} \sigma_{kl} = \sigma_{kl} \sigma_{ij} \text{ when } \{|i, j, k, l\} = 4\} = PAB$ .

**Theorem.** Let  $S = \langle \tau \rangle$  be the symmetric group. Then  $\mathfrak{B}$  is both  $PAB \rtimes S \cong B * S \langle \gamma_i \tau = \tau \gamma_i \text{ when } \tau i = j, \tau(i+1) = (j+1) \rangle$  (and so  $PAB$  is “bigger” than  $B$ , and hence quantum algebra doesn't see topology very well).

**Proof.** Going left,  $\gamma_i \mapsto \sigma_{i,i+1}(i+1)$ . Going right, if  $i < j$  map  $\sigma_{ij} \mapsto (j-1 \ j-2 \ \dots \ i) \gamma_{j-1}(i+1 \ \dots \ j)$  and if  $i > j$  use  $\sigma_{ij} \mapsto (j \ j+1 \ \dots \ i) \gamma_j(i-1 \ \dots \ j+1)$ .

**The Burau Representation** of  $PAB_n$  acts on  $\mathbb{R}^n := \mathbb{Z}[t^{\pm 1}]^n = R(v_1, \dots, v_n)$  by  $\sigma_{ij} v_k = v_k + \delta_{kj}(t-1)(v_j - v_i)$ .

**The Gassner Representation** of  $PAB_n$  acts on  $V = \mathbb{R}^n := \mathbb{Z}[t^{\pm 1}]^n = R(v_1, \dots, v_n)$  by  $\sigma_{ij} v_k = v_k + \delta_{kj}(t-1)(v_j - v_i)$ .

**The Turbo-Gassner Representation.** With the same  $R$  and  $V$ ,  $TG$  acts on  $V \oplus (R^n \oplus \mathbb{Q}) \oplus (S^2 V \oplus V^n) = R(v_i, u_i, u_{ij}, w_{ij})$  by  $TG_{i,j}[\mathcal{L}_i] := \mathcal{L}_i \cdot \{$   
 $v_i \mapsto v_i + \delta_{ij}((t-1)(v_j - v_i) + v_{i,j} - v_{i,i}) +$   
 $\delta_{ij}(u_{j,i} - u_{i,j}) u_{i,j}$   
 $v_{i,i} \mapsto v_{i,i} + (t-1) \cdot$   
 $(\delta_{i,j}(w_{i,j} - v_{i,i}) + (\delta_{i,i} - \delta_{i,j} t^2) t_j)$   
 $(u_i + \delta_{ij}(t-1)(u_j - u_i)) u_i w_{ij}$   
 $u_i \mapsto u_i + \delta_{ij}(t-1)(u_j - u_i)$   
 $w_{ij} \mapsto w_{ij} + (\delta_{i,j} - \delta_{j,i})(t^2 - 1) w_{ij}$  // Expand  
 $\text{bas3} = \{v_1, v_2, v_3, u_1^2 w_1, u_2^2 w_2, u_3^2 w_3, u_1 u_2 w_3, u_2 u_1 w_3, u_1 u_3 w_2, u_3 u_1 w_2, u_2 u_3 w_1, u_3 u_2 w_1, u_1 u_2 w_2, u_2 u_1 w_2, u_1 u_3 w_3, u_3 u_1 w_3, u_2 u_3 w_2, u_3 u_2 w_2\}$   
 $\text{bas3} // TG_{1,2} // TG_{1,3} // TG_{2,3} = (\text{bas3} // TG_{2,3} // TG_{1,2} // TG_{1,3})$   
True  
Like Gassner,  $TG$  is also a representation of  $PB_n$ .

**Identifiers:** Burau is a trivial silly reduction of Gassner.

**Gassner motifs:** [ωεβ/kbh](http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/)

**Adjoint-Gassner**

**My talk tomorrow, at the chord diagrams everywhere session:** [ωεβ/talks](http://www.math.toronto.edu/~drorbn/Talks/Toronto-1912/)

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