

**Kontsevich in a Pole Dance Studio.** (w/o poles? See [Ko, BN])

$$Z = \left( \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \sum_{\substack{I_1 < \dots < I_m \\ P = \{(z_i, z'_i)\}}} (-1)^{\#P_1} D_P \bigwedge_{i=1}^m \frac{dz_i - dz'_i}{z_i - z'_i} \right) \in \mathcal{A}$$

graded by the number of chords  
filtered by the number of ss chords

**Comments on the Kontsevich Integral.**

1. In the tangle case, the endpoints are fixed at top and bottom.
2. The  $(\dots)^\sim$  means “a correction is needed near the caps and the cups” (for the framed version, see [LM2, Da]).
3. There are never  $pp$  chords, and no  $4T_{pps}$  and  $4T_{ppp}$  relations.
4.  $Z$  is an “expansion”.
5.  $Z$  respects the  $ss$  filtration and so descends to  $Z^/s: \mathcal{K}^/s \rightarrow \mathcal{A}^/s$ .

**Comments on  $\mathcal{A}$ .** In  $\mathcal{A}^/1$  legs on poles commute, so  $\mathcal{A}^/1(\bigcirc) = |A|!$

In  $\mathcal{A}_H^/2$  we have:

**Example 1<sup>a</sup>.**  $\eta_1^a(|xyxy|, |xyx|) =$

**Example 3<sup>a</sup>.** Ignoring complications,  $\eta_3^a(xxyxyx) =$

**Proof of Lemma 1.** We partially prove Theorem 2 instead:  
**Theorem 2.**  $\text{gr}^\bullet \mathcal{K}_H \cong \mathbb{F}[[\hbar]] \otimes (\mathcal{K}^/1)_0$ .  
**Proof mod  $\hbar^2$ .** The map  $\leftarrow$  is obvious. To go  $\rightarrow$ , map  $\mathcal{K}_H \rightarrow \mathbb{F}[[\hbar]] \otimes \mathcal{K}^/1$  using  $\nearrow \mapsto \nearrow + \frac{\hbar}{2} \zeta$  and  $\searrow \mapsto \searrow - \frac{\hbar}{2} \zeta$  and apply the functor  $\text{gr}^\bullet$ .

**Unignoring the Complications.** We need  $\lambda_0$  and  $\lambda_1$  such that:

1.  $\lambda_1(\gamma)$  is obtained from  $\lambda_0(\gamma)$  by flipping all self-intersections from ascending to descending.
2. Up to conjugation,  $\lambda_1(\gamma)$  is obtained from  $\lambda_0(\gamma)$  by a global flip.
3.  $Z(\lambda_i(\gamma))$  is computable from  $W(\gamma)$  and  $Z^/1(\lambda_i(\gamma)) = W(\gamma)$ .

*Knitting needles*  
*Yarn*

1. Is there more than Examples 1–4? **Homework**
2. Derive the bialgebra axioms from this perspective.
3. What more do we get if we don't mod out by HOMFLY-PT?
4. What more do we get if we allow more than one strand-strand interaction?
5. In this language, recover Kashiwara-Vergne [AKKN1, AKKN2].
6. How is all this related to w-knots?
7. Do the same with associators. Use that to derive formulas for solutions of Kashiwara-Vergne.
8. What's the relationship with the Habiro-Massuyeau invariants of links in handlebodies [HM] (different filtration!).
9. Pole dance on other surfaces!
10. Explore the action of the mapping class group.

**Acknowledgement.** This work was partially supported by NSERC grant RGPIN-2018-04350 and by the Chu Family Foundation (NYC). I also wish to thank A. Alekseev, F. Naef, and M. Ren for listening to an earlier version and catching some bugs, and Dhanya S. for the dance studio photos. And of course, **thanks for listening!**

**References**

[AKKN1] A. Alekseev, N. Kawazumi, Y. Kuno, & F. Naef, *The Goldman-Turaev Lie Bialgebra in Genus Zero and the Kashiwara-Vergne Problem*, Adv. Math. **326** (2018) 1–53, arXiv:1703.05813.

[AKKN2] A. Alekseev, N. Kawazumi, Y. Kuno, & F. Naef, *Goldman-Turaev formality implies Kashiwara-Vergne*, Quant. Topol. **11-4** (2020) 657–689, arXiv:1812.01159.

[AN1] A. Alekseev & F. Naef, *Goldman-Turaev Formality from the Knizhnik-Zamolodchikov Connection*, Comp. Rend. Math. **355-11** (2017) 1138–1147, arXiv:1708.03119.

[BN] D. Bar-Natan, *On the Vassiliev Knot Invariants*, Top. **34** (1995) 423–472.

[Da] Z. Dancso, *On the Kontsevich Integral for Knotted Trivalent Graphs*, Alg. Geom. Topol. **10** (2010) 1317–1365, arXiv:0811.4615.

[HM] K. Habiro & G. Massuyeau, *The Kontsevich Integral for Bottom Tangles in Handlebodies*, Quant. Topol. **12-4** (2021) 593–703, arXiv:1702.00830.

[Ko] M. Kontsevich, *Vassiliev's Knot Invariants*, Adv. in Sov. Math. **16(2)** (1993) 137–150.

[LM1] T. Q. T. Le & J. Murakami, *Kontsevich's Integral for the HOMFLY Polynomial and Relations Between Values of Multiple Zeta Functions*, Top. and its Appl. **62-2** (1995) 193–206.

[LM2] T. Q. T. Le & J. Murakami, *The Universal Vassiliev-Kontsevich Invariant for Framed Oriented Links*, Comp. Math. **102-1** (1996) 41–64, arXiv: hep-th/9401016.

[Ma] G. Massuyeau, *Formal Descriptions of Turaev's Loop Operations*, Quant. Topol. **9-1** (2018) 39–117, arXiv:1511.03974.