$$
\begin{aligned}
dm_{i,j\to k} &\to a_k\,(\alpha_i+\alpha_j)+b_k\,(\beta_i+\beta_j)+y_k\,\eta_i+\frac{y_k\,\eta_j}{\mathcal{A}_i}+\frac{x_k\,\xi_i}{\mathcal{A}_j}+\eta_j\,\xi_i-\\
&\quad B_k\,\eta_j\,\xi_i+\frac{1}{4\,\mathcal{A}_i\,\mathcal{A}_j}\in\Big(2\,y_k\,\eta_j\,\big(2\,x_k\,\xi_i+\mathcal{A}_j\,(-2\,\beta_i+(1-3\,B_k)\,\eta_j\,\xi_i)\big)+\\
&\qquad \mathcal{A}_i\,\xi_i\,\Big(x_k\,(-4\,\beta_j+2\,(1-3\,B_k)\,\eta_j\,\xi_i)+\\
&\qquad\quad \mathcal{A}_j\,\eta_j\,\big(4\,a_k\,B_k+(1-4\,B_k+3\,B_k^2)\,\eta_j\,\xi_i\big)\Big)\Big)+x_k\,\xi_j\\
d\Delta_{i\to j,k} &\to a_j\,\alpha_i+a_k\,\alpha_i+b_j\,\beta_i+b_k\,\beta_i+y_j\,\eta_i+B_j\,y_k\,\eta_i+\\
&\quad x_j\,\xi_i+x_k\,\xi_i+\frac{1}{2}\in\big(B_j\,y_j\,y_k\,\eta_i^2+x_k\,\xi_i\,(-2\,a_j+x_j\,\xi_i)\big)\\
dS_i &\to -a_i\,\alpha_i-b_i\,\beta_i-\frac{\mathcal{A}_i\,(y_i\,\eta_i+(-\eta_i+B_i\,(x_i+\eta_i))\,\xi_i)}{B_i}-\\
&\quad \frac{1}{4\,B_i^2}\in\mathcal{A}_i\,\Big(\mathcal{A}_i\,\eta_i^2\,\big(2\,y_i^2-6\,y_i\,\xi_i+3\,\xi_i^2\big)+B_i^2\,\xi_i\,\big(4\,a_i\,x_i+2\,x_i^2\,\mathcal{A}_i\,\xi_i+\\
&\qquad\quad 2\,x_i\,(2\,\beta_i+\mathcal{A}_i\,\eta_i\,\xi_i)+\eta_i\,(-4+4\,\beta_i+\mathcal{A}_i\,\eta_i\,\xi_i)\big)+\\
&\qquad 2\,B_i\,\eta_i\,\big(y_i\,(-2+2\,\beta_i+2\,x_i\,\mathcal{A}_i\,\xi_i+\mathcal{A}_i\,\eta_i\,\xi_i)-\\
&\qquad\quad \xi_i\,(-2+2\,a_i+2\,\beta_i+3\,x_i\,\mathcal{A}_i\,\xi_i+2\,\mathcal{A}_i\,\eta_i\,\xi_i)\big)\Big)\\
R_{i,j} &\to a_j\,b_i+x_j\,y_i-\frac{1}{4}\in x_j^2\,y_i^2\\
P_{i,j} &\to \alpha_j\,\beta_i+\eta_i\,\xi_j+\frac{1}{4}\in\eta_i^2\,\xi_j^2
\end{aligned}
$$

A Quantum Algebra Example. ωεβ/qa

Proto-Proposition†0 (with Jesse Frohlich and Roland van der Veen, near [Ma, Proposition 1.7.3]). Let H be a finite dimensional Hopf algebra and let $U = H^{*cop} \otimes H$ be its Drinfel'd double, with R-matrix $R \in H^* \otimes H \subset U \otimes U$. Write $R^{\dagger 1} = \sum \rho_a \otimes r_a$, and let $\langle\cdot \mid \cdot\rangle\colon H^* \otimes H \to \mathbb{F}$ be the duality pairing. Then the functional $\int \in U^*$ defined by

$$\int \phi \otimes x := \sum \langle \phi \rho_a^{\dagger 2} \mid x r_a^{\dagger 3} \rangle$$

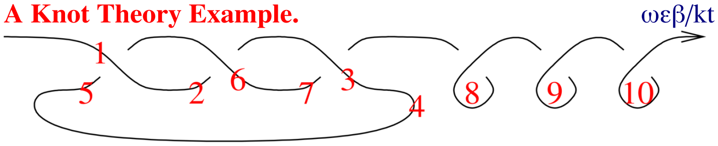
is a right†4 integral in U^*. (Meaning $\Delta^i_{jk} /\!\!/ \int_j = \int_i /\!\!/ \epsilon_k$ in $\mathrm{Hom}(U^{\otimes\{i\}} \to U^{\otimes\{k\}})$).

†0 A "proto-proposition" is something that will become a proposition once you figure out the correct statement. †1 Or did we want it to be $R/\!\!/S_1^2$? Or $R/\!\!/S_2^2$? †2 Or is it $\rho_a\phi$? †3 Or is it $r_a x$? †4 Or maybe "left"?

$$
\begin{aligned}
&\mathtt{inp} = \mathbb{E}_{\{\}\to\{1\}}[3\,a_1\,b_1,\ 5\,x_1\,y_1,\ 1]\ /\!/\ dm_{i,1\to i};\\
&\mathtt{Table}[\\
&\quad \mathtt{HL@TrueQ}[\\
&\qquad (\mathtt{inp}\ /\!/\ (s\Upsilon_{i\to 1,1,2,2}\,\mathtt{RR})\ /\!/\ \mathtt{BM}\ /\!/\ \mathtt{AM}\ /\!/\ P_{1,2})\ d\epsilon_j \equiv\\
&\qquad\ (\mathtt{inp}\ /\!/\ \Delta\Delta\ /\!/\ (s\Upsilon_{i\to 1,1,2,2}\,\mathtt{RR})\ /\!/\ \mathtt{BM}\ /\!/\ \mathtt{AM}\ /\!/\ P_{1,2})],\\
&\quad \{\Delta\Delta,\ \{d\Delta_{i\to i,j},\ d\Delta_{i\to j,i}\}\},\ \{\mathtt{AM},\ \{dm_{2,4\to 2},\ dm_{4,2\to 2}\}\},\\
&\quad \{\mathtt{BM},\ \{dm_{1,3\to 1},\ dm_{3,1\to 1}\}\},\\
&\quad \{\mathtt{RR},\ \{R_{3,4},\ R_{3,4}\ /\!/\ dS_3\ /\!/\ dS_3,\ R_{3,4}\ /\!/\ dS_4\ /\!/\ dS_4\}\}\\
&]\ /\!/\ \mathtt{MatrixForm}
\end{aligned}
$$

$$
\begin{pmatrix}
\begin{pmatrix} \mathrm{False} & \mathrm{False} & \mathrm{False} \\ \mathrm{False} & \mathrm{False} & \mathrm{False} \end{pmatrix} &
\begin{pmatrix} \mathrm{False} & \mathrm{False} & \mathrm{True} \\ \mathrm{False} & \mathrm{False} & \mathrm{False} \end{pmatrix} \\
\begin{pmatrix} \mathrm{False} & \mathrm{False} & \mathrm{False} \\ \mathrm{False} & \mathrm{False} & \mathrm{True} \end{pmatrix} &
\begin{pmatrix} \mathrm{False} & \mathrm{False} & \mathrm{False} \\ \mathrm{False} & \mathrm{False} & \mathrm{False} \end{pmatrix}
\end{pmatrix}
$$

A Knot Theory Example. ωεβ/kt



$$
\begin{aligned}
&\$k = 2;\\
&\mathtt{Simplify}\Big[\\
&\quad R_{1,5}\,R_{6,2}\,R_{3,7}\,\overline{C}_4\,\overline{\mathrm{Kink}}_8\,\overline{\mathrm{Kink}}_9\,\overline{\mathrm{Kink}}_{10}\ /\!/\ dm_{1,2\to 1}\ /\!/\ dm_{1,3\to 1}\ /\!/\\
&\qquad\quad dm_{1,4\to 1}\ /\!/\ dm_{1,5\to 1}\ /\!/\ dm_{1,6\to 1}\ /\!/\ dm_{1,7\to 1}\ /\!/\ dm_{1,8\to 1}\ /\!/\\
&\qquad dm_{1,9\to 1}\ /\!/\ dm_{1,10\to 1}\Big]\ /.\ \nu\_{}_1 :\to \nu
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{B}{1-B+B^2}+\\
&\quad \frac{B\,\big(-B+2\,B^2+2\,B^4+a\,(-1+B-B^3+B^4)-2\,x\,y-B^3\,(3+2\,x\,y)\big)\in}{(1-B+B^2)^3}+\\
&\quad \frac{1}{2\,(1-B+B^2)^5}\\
&\quad\ B\,\Big(4\,B^8+a^2\,(1-B+B^2)^2\,(1+B-6\,B^2+B^3+B^4)+6\,B^5\,x^2\,y^2+\\
&\qquad 2\,x\,y\,(-2+3\,x\,y)-B^7\,(11+4\,x\,y)-2\,B^2\,(1+6\,x^2\,y^2)-\\
&\qquad 2\,B^4\,(1-2\,x\,y+6\,x^2\,y^2)+B\,(1+8\,x\,y+6\,x^2\,y^2)+\\
&\qquad B^6\,(6+8\,x\,y+6\,x^2\,y^2)+B^3\,(4+4\,x\,y+30\,x^2\,y^2)+\\
&\qquad 2\,a\,(1-B+B^2)\,\big(2\,B^6+2\,x\,y+8\,B^3\,(1+x\,y)-5\,B^2\,(1+2\,x\,y)-\\
&\qquad\quad 2\,B^5\,(1+2\,x\,y)-B^4\,(7+2\,x\,y)+B\,(2+4\,x\,y)\big)\Big)\in^2+O[\in]^3\Big]
\end{aligned}
$$

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