Axioms. One axiom is primary and interesting,

- Contractions commute! Namely, $c_{x,\xi} /\!\!/ c_{y,\eta} = c_{y,\eta} /\!\!/ c_{x,\xi}$ (or in old-speak, $c_{y,\eta} \circ c_{x,\xi} = c_{x,\xi} \circ c_{y,\eta}$).

And the rest are just what you'd expect:

- $\sqcup$ is commutative and associative, and it commutes with $c_{\cdot,\cdot}$ and with $\sigma^{\cdot}_{\cdot}$ whenever that makes sense.
- $c_{\cdot,\cdot}$ is "natural" relative to renaming: $c_{x,\xi} = \sigma^x_y /\!\!/ \sigma^\xi_\eta /\!\!/ c_{y,\eta}$.
- $\sigma^\xi_\xi = \sigma^x_x = Id$, $\sigma^\xi_\eta /\!\!/ \sigma^\eta_\zeta = \sigma^\xi_\zeta$, $\sigma^x_y /\!\!/ \sigma^y_z = \sigma^x_z$, and renaming operations commute where it makes sense.

Comments.

- We can relax $|\mathcal{X}| = |X|$ at no cost.
- We can lose the distinction between $\mathcal{X}$ and X and get "circuit algebras".
- There is a "coloured version", where $\mathcal{T}(\mathcal{X}, X)$ is replaced with $\mathcal{T}(\mathcal{X}, X, \lambda, l)$ where $\lambda \colon \mathcal{X} \to C$ and $l \colon X \to C$ are "colour functions" into some set C of "colours", and contractions $c_{x,\xi}$ are allowed only if x and ξ are of the same colour, $l(x) = \lambda(\xi)$. In the world of tangles, this is "coloured tangles".

2. Heaven is a Place on Earth

(A version of the main results of Archibald's thesis, [Ar]).

Let us work over the base ring $\mathcal{R} = \mathbb{Q}[\{T^{\pm 1/2} \colon T \in C\}]$. Set

$$\mathcal{A}(\mathcal{X}, X) := \{w \in \Lambda(\mathcal{X} \sqcup X) \colon \deg_{\mathcal{X}} w = \deg_X w\}$$

(so in particular the elements of $\mathcal{A}(\mathcal{X}, X)$ are all of even degree). The union operation is the wedge product, the renaming operations are changes of variables, and $c_{x,\xi}$ is defined as follows. Write $w \in \mathcal{A}(\mathcal{X}, X)$ as a sum of terms of the form uw' where $u \in \Lambda(\xi, x)$ and $w' \in \mathcal{A}(\mathcal{X} \setminus \xi, X \setminus x)$, and map u to 1 if it is 1 or $x\xi$ and to 0 is if is ξ or x:

$$1w' \mapsto w', \qquad \xi w' \mapsto 0, \qquad xw' \mapsto 0, \qquad x\xi w' \mapsto w'.$$

Proposition. $\mathcal{A}$ is a contraction algebra.

Alternative Formulations.

- $$c_{x,\xi} w = \iota_\xi \iota_x \mathbb{e}^{x\xi} w,$$ where $\iota.$ denotes interior multiplication.
- Using Fermionic integration, $$c_{x,\xi} w = \int \mathbb{e}^{x\xi} w \, d\xi dx.$$
- $c_{x,\xi}$ represents composition in exterior algebras! With $X^* := \{x^* \colon x \in X\}$, we have that $\operatorname{Hom}(\Lambda X, \Lambda Y) \cong \Lambda(X^* \sqcup Y)$ and the following square commutes:

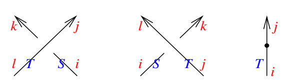
$$\begin{tikzcd}
\operatorname{Hom}(\Lambda X, \Lambda Y) \otimes \operatorname{Hom}(\Lambda Y, \Lambda Z) \arrow[r, "/\!\!/"] \arrow[d, leftrightarrow] & \operatorname{Hom}(\Lambda X, \Lambda Z) \arrow[d, leftrightarrow] \\
\Lambda(X^* \sqcup Y \sqcup Y^* \sqcup Z) \arrow[r, "\prod_{y \in Y} c_{y,y^*}"] & \Lambda(X^*, Z)
\end{tikzcd}$$

- Similarly, $\Lambda(\mathcal{X} \sqcup X) \cong (H^*)^{\otimes \mathcal{X}} \otimes H^{\otimes X}$ where H is a 2-dimensional "state space" and H^* is its dual. Under this identification, $c_{x,\xi}$ becomes the contraction of an H factor with an H^* factor.

We construct a morphism of coloured contraction algebras $\mathcal{A} \colon \mathcal{T} \to \mathcal{A}$ by declaring

$$\begin{aligned}
X_{ijkl}[S, T] &\mapsto T^{-1/2} \exp\left(\begin{pmatrix} \xi_l & \xi_i \end{pmatrix} \begin{pmatrix} 1 & 1-T \\ 0 & T \end{pmatrix} \begin{pmatrix} x_j \\ x_k \end{pmatrix} \right) \\
\bar{X}_{ijkl}[S, T] &\mapsto T^{1/2} \exp\left(\begin{pmatrix} \xi_i & \xi_j \end{pmatrix} \begin{pmatrix} T^{-1} & 0 \\ 1-T^{-1} & 1 \end{pmatrix} \begin{pmatrix} x_k \\ x_l \end{pmatrix} \right) \\
P_{ij}[T] &\mapsto \exp(\xi_i x_j)
\end{aligned}$$

with



$X_{ijkl}[S, T]$ $\qquad$ $\bar{X}_{ijkl}[S, T]$ $\qquad$ $P_{ij}[T]$

(Note that the matrices appearing in these formulas are the Burau matrices).

Theorem.

If D is a classical link diagram with k components coloured $T_1, \ldots, T_k$ whose first component is open and the rest are closed, if MVA is the multivariable Alexander polynomial of the closure of D (with these colours), and if ρ_j is the counterclockwise rotation number of the jth component of D, then

$$\mathcal{A}(D) = T_1^{-1/2}(T_1 - 1) \left(\prod_j T_j^{\rho_j/2} \right) \cdot MVA \cdot (1 + \xi_{\text{in}} \wedge x_{\text{out}}).$$

($\mathcal{A}$ vanishes on closed links).

3. An Implementation of $\mathcal{A}$

If I didn't implement I wouldn't believe myself.

Written in Mathematica [Wo], available as the notebook `Alpha.nb` at http://drorbn.net/mo21/ap. Code lines are highlighted in grey, demo lines are plain. We start with an implementation of elements ("`Wedge`") of exterior algebras, and of the wedge product ("`WP`"):

```
WP[Wedge[u___], Wedge[v___]] := Signature[{u, v}] * Wedge @@ Sort[{u, v}];
WP[0, _] = WP[_, 0] = 0;
WP[A_, B_] :=
  Expand[Distribute[A ** B] /.
    (a_. * u_Wedge) ** (b_. * v_Wedge) :→ a b WP[u, v]];
```

```
WP[Wedge[‸] + Wedge[a] - 2 b ∧ a, Wedge[‸] - 3 Wedge[b] + 7 c ∧ d]
```

```
Wedge[] + Wedge[a] - 3 Wedge[b] - a ∧ b + 7 c ∧ d + 7 a ∧ c ∧ d + 14 a ∧ b ∧ c ∧ d
```

We then define the exponentiation map in exterior algebras ("`WExp`") by summing the series and stopping the sum once the current term ("`t`") vanishes:

```
WExp[A_] := Module[{s = Wedge[‸], t = Wedge[‸], k = 0},
  While[t =!= 0, s += (t = Expand[WP[t, A] / (++k)])]; s]
```

```
WExp[a ∧ b + c ∧ d + e ∧ f]
```

```
Wedge[] + a ∧ b + c ∧ d + e ∧ f + a ∧ b ∧ c ∧ d + a ∧ b ∧ e ∧ f + c ∧ d ∧ e ∧ f + a ∧ b ∧ c ∧ d ∧ e ∧ f
```

Video and more at http://www.math.toronto.edu/~drorbn/Talks/MoscowByWeb-2104/