The effort to take a single multi-bite is tiny. Indeed,

Lemma Given $2d$ finite sets $B_i = \{t_{i1}, t_{i2}, \dots\} \subset [1..L^3]$ and a permutation $\pi \in S_{2n}$ the quantity

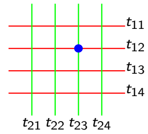
$$N = \left| \left\{ (b_i) \in \prod_{i=1}^{2d} B_i \colon \text{the } b_i\text{'s are ordered as } \pi \right\} \right|$$

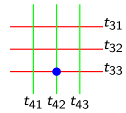
can be computed in time $\sim \sum |B_i| \sim \max |B_i|$.

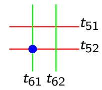
Proof. WLOG $\pi = Id$. For $\iota \in [1..2d]$ and $\beta \in B \coloneqq \cup B_i$ let

$$N_{\iota,\beta} = \left| \left\{ (b_i) \in \prod_{i=1}^{\iota} B_i \colon b_1 < b_2 < \ldots < b_\iota \leq \beta \right\} \right|.$$

We need to know $N_{2d,\max B}$; compute it inductively using $N_{\iota,\beta} = N_{\iota,\beta'} + N_{\iota-1,\beta'}$, where β' is the predecessor of β in B. $\square$







The two methods agree (and therefore are at their worst) if $2^G = L^{\frac{4}{7}(d-1)}$, and in that case, they both take time $\sim L^{\frac{18}{7}d+\frac{3}{7}} = V^{\frac{6}{7}d+\frac{1}{7}}$.

The same reasoning, with the $n^{(\frac{2}{3}+\epsilon)d}$ feather, gives $V^{(\frac{4}{5}+\epsilon)d}$.

$\square$

Conclusion. We wish to compute the contribution to φ_d coming from d-tuples of crossings of multi-generation $\bar{g}$.

- The multi-shark method does it in time

$$\sim (\text{no. of bites}) \cdot (\text{time per bite}) = L^{2d} 2^G \cdot \frac{L}{2^{\min \bar{g}}} < L^{2d+1} 2^G$$

(increases with G).

- The multi-feather method (project and use the 2D algorithm) does it in time

$$\sim (\text{no. of crossings})^{\lfloor \frac{3}{4} d \rfloor} = \left(\prod_{i=1}^{d} L^2 \frac{L^2}{2^{g_i}} \right)^{\lfloor \frac{3}{4} d \rfloor} < \frac{L^{3d}}{(2^G)^{3/4}}$$

(decreases with G).

Of course, for any specific G we are free to choose whichever is better, shark or feather.

If time — a word about braids.

Thank You!