Note.

The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2}\det(A),$$

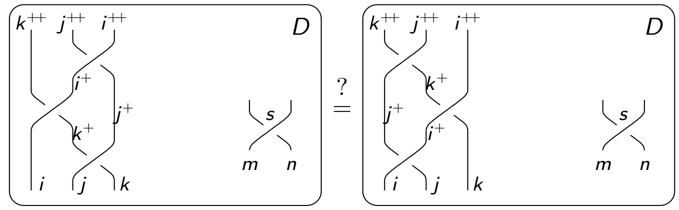
with

$$\varphi = \sum_k \varphi_k, \qquad w = \sum_c s.$$

We also set $\Delta_\nu := \Delta(T_\nu)$ for $\nu = 1, 2, 3$. This defines and explains the normalization factors in the Main Theorem.

Corollary 2.

Proving invariance is easy:



Invariance under R3

This is Theta.nb of http://drorbn.net/ktc25/ap.

```
Once[<< KnotTheory`; << Rot.m; << PolyPlot.m];
Loading KnotTheory` version of October 29, 2024, 10:29:52.1301.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/ktc25/ap to compute rotation numbers.
Loading PolyPlot.m from
  http://drorbn.net/ktc25/ap to plot 2-variable polynomials.
T3 = T1 T2;
CF[ℰ_] := Expand@Collect[ℰ, g__, F] /. F → Factor;
```

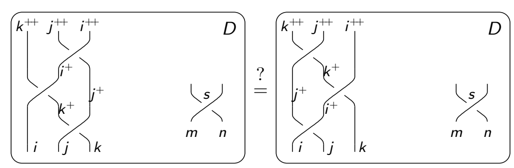
ωεβ:=http://drorbn.net/ktc25

```
F1[{s_, i_, j_}] =
  CF[
   s (1/2 - g3ii + T2^s g1ii g2ji - g1ii g2jj - (T2^s - 1) g2ji g3ii + 2 g2jj g3ii -
      (1 - T3^s) g2ji g3ji - g2ii g3jj - T2^s g2ji g3jj + g1ii g3jj +
      ((T1^s - 1) g1ji (T2^(2 s) g2ji - T2^s g2jj + T2^s g3jj) +
         (T3^s - 1) g3ji (1 - T2^s g1ii - (T1^s - 1) (T2^s + 1) g1ji + (T2^s - 2) g2jj + g2ij)) /
       (T2^s - 1))];
F2[{s0_, i0_, j0_}, {s1_, i1_, j1_}] :=
 CF[s1 (T1^s0 - 1) (T2^s1 - 1)^-1 (T3^s1 - 1) g1,j1,i0 g3,j0,i1
   ((T2^s0 g2,i1,i0 - g2,i1,j0) - (T2^s0 g2,j1,i0 - g2,j1,j0))]
F3[φ_, k_] = -φ/2 + φ g3kk;
```

ωεβ:=http://drorbn.net/ktc25

```
δi_,j_ := If[i === j, 1, 0];
gRs_,i_,j_ := {
   gν_jβ_ :→ gνj+β + δjβ, gν_iβ_ :→ Tν^s gνi+β + (1 - Tν^s) gνj+β + δiβ,
   gν_α_i+ :→ Tν^s gναi + δαi+, gν_α_j+ :→ gναj + (1 - Tν^s) gναi + δαj+
  }
```

ωεβ:=http://drorbn.net/ktc25




```
DSum[Cs___] := Sum[F1[c], {c, {Cs}}] +
  Sum[F2[c0, c1], {c0, {Cs}}, {c1, {Cs}}]
lhs = DSum[{1, j, k}, {1, i, k+}, {1, i+, j+}, {s, m, n}] //.
   gR1,j,k ⋃ gR1,i,k+ ⋃ gR1,i+,j+;
rhs = DSum[{1, i, j}, {1, i+, k}, {1, j+, k+}, {s, m, n}] //.
   gR1,i,j ⋃ gR1,i+,k ⋃ gR1,j+,k+;
Simplify[lhs == rhs]
True
```

ωεβ:=http://drorbn.net/ktc25

The Main Program

```
Θ[K_] := Module[{Cs, φ, n, A, Δ, G, ev, θ},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :→ (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1
                                                              0     -1   ))];
  Δ = T^((-Total[φ] - Total[Cs[[All, 1]]])/2) Det[A];
  G = Inverse[A];
  ev[ℰ_] := Factor[ℰ /. gν_,α_,β_ :→ (G[[α, β]] /. T → Tν)];
  θ = ev[Σ_{k=1}^{n} F1[Cs[[k]]]];
  θ += ev[Σ_{k1=1}^{n} Σ_{k2=1}^{n} F2[Cs[[k1]], Cs[[k2]]]];

  θ += ev[Σ_{k=1}^{2 n} F3[φ[[k]], k]];
  Factor@{Δ, (Δ /. T → T1) (Δ /. T → T2) (Δ /. T → T3) θ}];
```

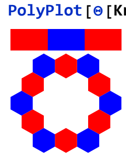
ωεβ:=http://drorbn.net/ktc25

The Trefoil Knot

```
Θ[Knot[3, 1]] // Expand
```

$$\left\{-1+\frac{1}{T}+T,\ -\frac{1}{T_1^2}-T_1^2-\frac{1}{T_2^2}-\frac{1}{T_1^2\,T_2^2}+\frac{1}{T_1\,T_2^2}+\frac{1}{T_1^2\,T_2}+\frac{T_1}{T_2}+\frac{T_2}{T_1}+T_1^2\,T_2-T_2^2+T_1\,T_2^2-T_1^2\,T_2^2\right\}$$

```
PolyPlot[Θ[Knot[3, 1]], ImageSize → Tiny]
```



Video and more at http://www.math.toronto.edu/~drorbn/Talks/KnotTheoryCongress-2502.