

$\mathcal{D}: \text{Hom}(U^{\otimes \Sigma} \rightarrow U^{\otimes S}) \rightarrow \mathbb{Q}[\eta_\Sigma, \beta_\Sigma, \alpha_\Sigma, \xi_\Sigma, \gamma_S, b_S, a_S, x_S]$ . The PBW theorem for  $CU$  (always in the  $ybax$  order), or its quantum analog for  $QU$ , say that if  $U = CU$  or  $QU$  then  $U^{\otimes S}$  is isomorphic as a vector space to  $\mathbb{Q}[y_i, b_i, a_i, x_i]_{i \in S}[[\hbar]]$ ; so it is enough to understand  $\text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B])$  for finite sets  $A$  and  $B$ .

**Claim.**  $F \in \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B]) \xrightarrow{\mathcal{D}} \mathbb{Q}[z_B][[\zeta_A]] \ni \mathcal{F}$  via

$$\mathcal{D}(F) := \sum_{n \in \mathbb{N}^A} \frac{\zeta_A^n}{n!} F(z_A^n) = F\left(\bigoplus_{a \in A} \zeta_a z_a\right) = \mathcal{F},$$

$$\mathcal{D}^{-1}(\mathcal{F})(p) = \left( p|_{z_a \rightarrow \partial_{z_a} \mathcal{F}} \right)_{\zeta_a=0} \quad \text{for } p \in \mathbb{Q}[z_A].$$

**Claim.** Assuming convergence, if  $F \in \text{Hom}(\mathbb{Q}[z_A] \rightarrow \mathbb{Q}[z_B])$ ,  $G \in \text{Hom}(\mathbb{Q}[z_B] \rightarrow \mathbb{Q}[z_C])$ ,  $\mathcal{F} = \mathcal{D}(F)$ , and  $\mathcal{G} = \mathcal{D}(G)$ , then

$$\mathcal{D}(F \circ G) = \left( \mathcal{F}|_{z_i \rightarrow \partial_{z_i} \mathcal{G}} \right)_{\zeta_i=0}.$$

And so the title of the talk finally makes sense!

**Example.**  $\mathcal{D}(id: U \rightarrow U) = \mathbb{e}^{\eta y + \beta b + \alpha a + \xi x}$ .

**Example.** Let  $c\Delta_{jk}^i: CU^{\otimes \{i\}} \rightarrow CU^{\otimes \{j,k\}}$  be the standard co-product, given by  $c\Delta_{jk}^i(y_i, b_i, a_i, x_i) = (y_j + y_k, b_j + b_k, a_j + a_k, x_j + x_k)$ . Then

$$\begin{aligned} \mathcal{D}(c\Delta_{jk}^i) &= c\Delta_{jk}^i(\mathbb{e}^{\eta_i y_i + \beta_i b_i + \alpha_i a_i + \xi_i x_i}) \\ &= \mathbb{e}^{\eta_i(y_j + y_k) + \beta_i(b_j + b_k) + \alpha_i(a_j + a_k) + \xi_i(x_j + x_k)}. \end{aligned}$$

**Example.** The standard commutative product  $m_k^{ij}$  of polynomials is given by  $z_i, z_j \rightarrow z_k$ . Hence  $\mathcal{D}(m_k^{ij}) = m_k^{ij}(\mathbb{e}^{\zeta_i z_i + \zeta_j z_j}) = \mathbb{e}^{(\zeta_i + \zeta_j) z_k}$ .

$$\begin{array}{ccc} \mathbb{Q}[z_i] \otimes \mathbb{Q}[z_j] & \xrightarrow{m_k^{ij}} & \mathbb{Q}[z_k] \\ \parallel & & \parallel \\ \mathbb{Q}[z_i, z_j] & \xrightarrow{m_k^{ij}} & \mathbb{Q}[z_k] \end{array}$$

**A real DoPeGDO Example.** Let  $cm_k^{ij}: CU_i \otimes CU_j \rightarrow CU_k$  be ‘‘classical multiplication’’ for  $sl_{2+}^\xi$ , and let  $\mathbb{O}_i: \mathbb{Q}[y_i, b_i, a_i, x_i] \rightarrow CU_i$  be the PBW ordering map.

$$\begin{array}{ccc} CU_i \otimes CU_j & \xrightarrow{cm_k^{ij}} & CU_k \\ \uparrow \mathbb{O}_{i,j} & & \uparrow \mathbb{O}_k \\ \mathbb{Q}[y_i, b_i, a_i, x_i, y_j, b_j, a_j, x_j] & & \mathbb{Q}[y_k, b_k, a_k, x_k] \end{array}$$

**Claim.** Let

$$\begin{aligned} \Lambda &= \left( \eta_i + \frac{e^{-\alpha_i - \epsilon \beta_i} \eta_j}{1 + \epsilon \eta_j \xi_i} \right) y_k + \left( \beta_i + \beta_j + \frac{\log(1 + \epsilon \eta_j \xi_i)}{\epsilon} \right) b_k + \\ &\quad (\alpha_i + \alpha_j + \log(1 + \epsilon \eta_j \xi_i)) a_k + \left( \frac{e^{-\alpha_j - \epsilon \beta_j} \xi_i}{1 + \epsilon \eta_j \xi_i} + \xi_j \right) x_k \end{aligned}$$

Then  $\mathbb{e}^{\eta_i y_i + \beta_i b_i + \alpha_i a_i + \xi_i x_i + \eta_j y_j + \beta_j b_j + \alpha_j a_j + \xi_j x_j} \circ \mathbb{O}_{i,j} \circ cm_k^{ij} = \mathbb{e}^\Lambda \circ \mathbb{O}_k$ , and hence  $\mathcal{D}(cm_k^{ij}) = \mathbb{e}^\Lambda$  and  $cm_k^{ij}$  is DoPeGDO.

**Proof.** We compute in a faithful 2D representation  $z \mapsto \hat{z}$  of  $CU$ :

$\text{HL}[\mathcal{E}_-] := \text{Style}[\mathcal{E}, \text{Background} \rightarrow \text{If}[\text{TrueQ}@\mathcal{E}, \blacksquare, \blacksquare]];$

$\{\hat{y} = \begin{pmatrix} \theta & \theta \\ \epsilon & \theta \end{pmatrix}, \hat{b} = \begin{pmatrix} \theta & \theta \\ \theta & -\epsilon \end{pmatrix}, \hat{a} = \begin{pmatrix} 1 & \theta \\ \theta & \theta \end{pmatrix}, \hat{x} = \begin{pmatrix} \theta & 1 \\ \theta & \theta \end{pmatrix}\};$

$\text{HL} / \{ \hat{a} \cdot \hat{x} - \hat{x} \cdot \hat{a} = \hat{x}, \hat{a} \cdot \hat{y} - \hat{y} \cdot \hat{a} = -\hat{y}, \hat{b} \cdot \hat{y} - \hat{y} \cdot \hat{b} = -\epsilon \hat{y},$

$\hat{b} \cdot \hat{x} - \hat{x} \cdot \hat{b} = \epsilon \hat{x}, \hat{x} \cdot \hat{y} - \hat{y} \cdot \hat{x} = \hat{b} + \epsilon \hat{a} \}$

$\{\text{True}, \text{True}, \text{True}, \text{True}, \text{True}\}$

$\text{HL}@\text{Simplify}@\text{With}[\{\mathbb{E} = \text{MatrixExp}\},$

$\mathbb{E}[\eta_i \hat{y}] \cdot \mathbb{E}[\beta_i \hat{b}] \cdot \mathbb{E}[\alpha_i \hat{a}] \cdot \mathbb{E}[\xi_i \hat{x}] \cdot \mathbb{E}[\eta_j \hat{y}] \cdot \mathbb{E}[\beta_j \hat{b}] \cdot$   
 $\mathbb{E}[\alpha_j \hat{a}] \cdot \mathbb{E}[\xi_j \hat{x}] = \mathbb{E}[\hat{y} \partial_{y_k} \Lambda] \cdot \mathbb{E}[\hat{b} \partial_{b_k} \Lambda] \cdot \mathbb{E}[\hat{a} \partial_{a_k} \Lambda] \cdot$   
 $\mathbb{E}[\hat{x} \partial_{x_k} \Lambda]$

**True**

**Series**  $[\Lambda, \{\epsilon, \theta, 1\}]$

$$\begin{aligned} &(\mathbf{a}_k (\alpha_i + \alpha_j) + \mathbf{y}_k (\eta_i + e^{-\alpha_i} \eta_j) + \\ &\quad \mathbf{b}_k (\beta_i + \beta_j + \eta_j \xi_i) + \mathbf{x}_k (e^{-\alpha_j} \xi_i + \xi_j)) + \\ &\left( \mathbf{a}_k \eta_j \xi_i - \frac{1}{2} \mathbf{b}_k \eta_j^2 \xi_i^2 - e^{-\alpha_i} \mathbf{y}_k \eta_j (\beta_i + \eta_j \xi_i) - \right. \\ &\quad \left. e^{-\alpha_j} \mathbf{x}_k \xi_i (\beta_j + \eta_j \xi_i) \right) \epsilon + \mathbf{O}[\epsilon]^2 \end{aligned}$$

(Shame, but this technique fails for  $QU$ ).

**Claim.** In  $QU$ ,  $R$  is DoPeGDO.

**Proof.** Recall that with  $q = e^{\hbar \epsilon}$ ,

$$R = \sum \hbar^{j+k} y^k b^j \otimes a^j x^k / j! k! q! = \mathbb{O} \left( \mathbb{e}^{\hbar b_1 a_2} \mathbb{e}^{\hbar y_1 x_2} \right).$$

Now expand  $\mathbb{e}^{\hbar y_1 x_2}$  in powers of  $\epsilon$  using:

**Faddeev’s Formula** (In as much as we can tell, first appeared without proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]).

With  $[n]_q := \frac{q^n - 1}{q - 1}$ , with  $[n]_q! := [1]_q [2]_q \cdots [n]_q$  and with  $\mathbb{e}_q^x := \sum_{n \geq 0} \frac{x^n}{[n]_q!}$ , we have

$$\log \mathbb{e}_q^x = \sum_{k \geq 1} \frac{(1 - q)^k x^k}{k(1 - q^k)} = x + \frac{(1 - q)^2 x^2}{2(1 - q^2)} + \dots$$

**Proof.** We have that  $\mathbb{e}_q^x = \frac{\mathbb{e}_q^{qx} - \mathbb{e}_q^x}{qx - x}$  (‘‘the  $q$ -derivative of  $\mathbb{e}_q^x$  is itself’’), and hence  $\mathbb{e}_q^{qx} = (1 + (1 - q)x)\mathbb{e}_q^x$ , and

$$\log \mathbb{e}_q^{qx} = \log(1 + (1 - q)x) + \log \mathbb{e}_q^x.$$

Writing  $\log \mathbb{e}_q^x = \sum_{k \geq 1} a_k x^k$  and comparing powers of  $x$ , we get  $q^k a_k = -(1 - q)^k / k + a_k$ , or  $a_k = \frac{(1 - q)^k}{k(1 - q^k)}$ .  $\square$

**Compositions (2).** Recall that with all indices  $i$  running in some set  $B$ ,

$$\mathcal{F} \circ \mathcal{G} = \left( \mathcal{F}|_{z_i \rightarrow \partial_{z_i} \mathcal{G}} \right)_{\zeta_i=0} \stackrel{(1)}{=} \mathbb{e}^{\sum \partial_{z_i} \partial_{z_i} (\mathcal{F} \mathcal{G})} \Big|_{z_i = \zeta_i = 0}, \quad \begin{array}{l} (1) \text{ Strictly speaking,} \\ \text{true only when} \\ B \cap (A \cup C) = \emptyset. \end{array}$$

so in general we wish to understand

$$[F: \mathcal{E}]_B := \mathbb{e}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} \partial_{z_i} \partial_{z_j} \mathcal{E}} \quad \text{and} \quad \langle F: \mathcal{E} \rangle_B := [F: \mathcal{E}]_B|_{z_B \rightarrow 0},$$

where  $\mathcal{E}$  is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where  $\mathcal{E}$  has no  $B$ - $B$  quadratic part:

**Lemma 1.** With convergences left to the reader,

$$\left\langle F: \mathcal{E} \mathbb{e}^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1}: \mathcal{E} \right\rangle_B.$$

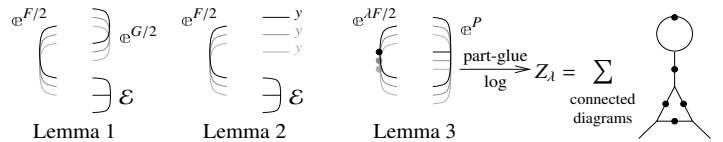
The next lemma dispatches the case where  $\mathcal{E}$  has a  $B$ -linear part:

**Lemma 2.**  $\left\langle F: \mathcal{E} \mathbb{e}^{\sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{e}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F: \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B$ .

Finally, we deal with the docile perturbation case:

**Lemma 3.** With an extra variable  $\lambda$ ,  $Z_\lambda := \log[\lambda F: \mathbb{e}^P]_B$  satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left( \partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$



**Complexity** to  $\epsilon^k$ , for an  $n$ -xing width  $w$  knot (by [LT],  $w \in O(\sqrt{n})$ ), is  $O(n^2 w^{2k+2} \log n) = O(n^{k+3} \log n)$  integer operations.