

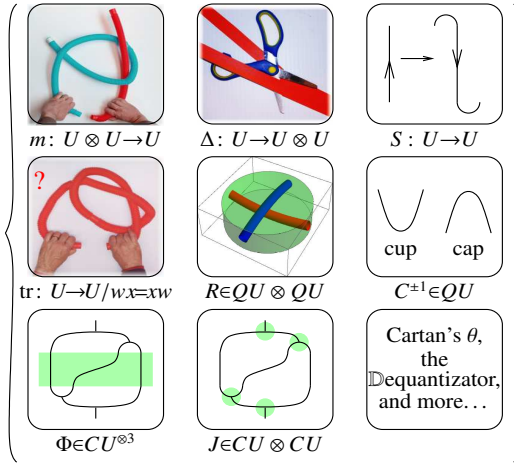


Everything around sl_{2+}^ϵ is DoPeGDO. So what?

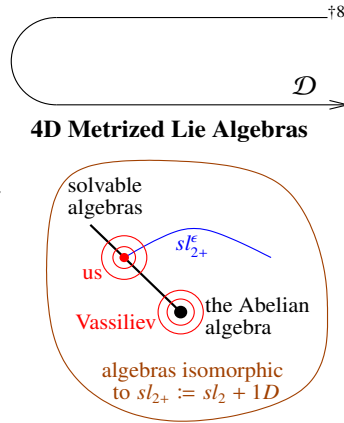
Abstract. I'll explain what "everything around" means: classical and quantum m, Δ, S, tr, R, C , and θ , as well as P, Φ, J, \mathbb{D} , and more, and all of their compositions. What **DoPeGDO** means: the category of **Docile Perturbed Gaussian Differential Operators**. And what sl_{2+}^ϵ means: a solvable approximation of the semi-simple Lie algebra sl_2 .

Knot theorists should rejoice because all this leads to very powerful and well-behaved poly-time-computable knot invariants. Quantum algebraists should rejoice because it's a realistic playground for testing complicated equations and theories.

Conventions. 1. For a set A , let $z_A := \{z_i\}_{i \in A}$ and let $\zeta_A := \{\zeta_i^* = \zeta_i\}_{i \in A}$.^{†1} 2. Everything converges!



Less Abstract



DoPeGDO := The category with objects finite sets^{†2} and $\text{mor}(A \rightarrow B)$:

$$\{\mathcal{F} = \omega \exp(Q + P)\} \subset \mathbb{Q}[\zeta_A, z_B, \epsilon]$$

Where: • ω is a scalar.^{†3} • Q is a "small" ϵ -free quadratic in $\zeta_A \cup z_B$.^{†4} • P is a "docile perturbation": $P = \sum_{k \geq 1} \epsilon^k P^{(k)}$, where $\text{deg } P^{(k)} \leq 2k + 2$.^{†5} • Compositions:^{†6}

$$\mathcal{F} // \mathcal{G} = \mathcal{G} \circ \mathcal{F} := (\mathcal{G}|_{\zeta_i \rightarrow \partial_{\zeta_i} \mathcal{F}})_{z_i=0} = (\mathcal{F}|_{z_i \rightarrow \partial_{\zeta_i} \mathcal{G}})_{\zeta_i=0}$$

Cool! $(V^*)^{\otimes \Sigma} \otimes V^{\otimes \Sigma}$ explodes; the ranks of quadratics and bounded-degree polynomials grow slowly!^{†7} **Representation theory is over-rated!**

Cool! How often do you see a computational toolbox so successful?

Our Algebras. Let $sl_{2+}^\epsilon := L\langle y, b, a, x \rangle$ subject to $[a, x] = x$, $[b, y] = -\epsilon y$, $[a, b] = 0$, $[a, y] = -y$, $[b, x] = \epsilon x$, and $[x, y] = \epsilon a + b$. So $t := \epsilon a - b$ is central and if $\exists \epsilon^{-1}$, $sl_{2+}^\epsilon / \langle t \rangle \cong sl_2$.^{ωεβ/oa} U is either $CU = \mathcal{U}(sl_{2+}^\epsilon)[[\hbar]]$ or $QU = \mathcal{U}_\hbar(sl_{2+}^\epsilon) = A\langle y, b, a, x \rangle[[\hbar]]$ with $[a, x] = x$, $[b, y] = -\epsilon y$, $[a, b] = 0$, $[a, y] = -y$, $[b, x] = \epsilon x$, and $xy - qyx = (1 - AB)/\hbar$, where $q = e^{\hbar \epsilon}$, $A = e^{-\hbar \epsilon a}$, and $B = e^{-\hbar b}$. Set also $T = A^{-1}B = e^{\hbar t}$.

The Quantum Leap. Also decree that in QU ,

$$\Delta(y, b, a, x) = (y_1 + B_1 y_2, b_1 + b_2, a_1 + a_2, x_1 + A_1 x_2),$$

$$S(y, b, a, x) = (-B^{-1} y, -b, -a, -A^{-1} x),$$

and $R = \sum \hbar^{j+k} y^j b^k x^j \otimes a^j x^k / j! [k]_q!$.

Mid-Talk Debts. • What is this good for in quantum algebra?

- In knot theory?
- How does the "inclusion" $\mathcal{D}: \text{Hom}(U^{\otimes \Sigma} \rightarrow U^{\otimes S}) \rightsquigarrow$ **DoPeGDO** work?
- Proofs that everything around sl_{2+}^ϵ really is **DoPeGDO**.
- Relations with prior art.
- The rest of the "compositions" story.

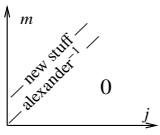
Theorem ([BG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K , in the d -dimensional representation of sl_2 . Writing

$$\left. \frac{(q^{1/2} - q^{-1/2}) J_d(K)}{q^{d/2} - q^{-d/2}} \right|_{q=e^\hbar} = \sum_{j,m \geq 0} a_{jm}(K) d^j \hbar^m,$$

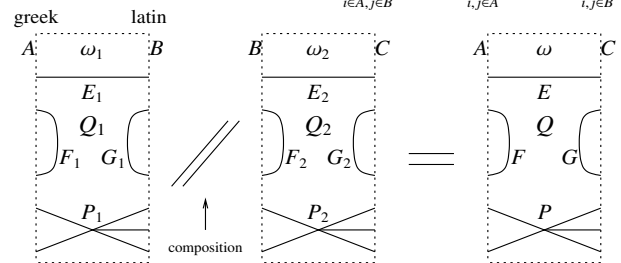
"below diagonal" coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and "on diagonal" coefficients give the inverse of the Alexander polynomial: $(\sum_{m=0}^\infty a_{mm}(K) \hbar^m) \cdot \omega(K)(e^\hbar) = 1$.

"Above diagonal" we have **Rozansky's Theorem** [Ro3, (1.2)]:

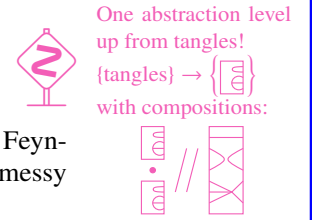
$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1}) \omega(K)(q^d)} \left(1 + \sum_{k=1}^\infty \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)} \right).$$



Compositions (1). In $\text{mor}(A \rightarrow B)$, $Q = \sum_{i \in A, j \in B} E_{ij} \zeta_i z_j + \frac{1}{2} \sum_{i, j \in A} F_{ij} \zeta_i \zeta_j + \frac{1}{2} \sum_{i, j \in B} G_{ij} z_i z_j$



- $E = E_1(I - F_2 G_1)^{-1} E_2$.
- $F = F_1 + E_1 F_2 (I - G_1 F_2)^{-1} E_1^T$.
- $G = G_2 + E_2^T G_1 (I - F_2 G_1)^{-1} E_2$.
- $\omega = \omega_1 \omega_2 \det(I - F_2 G_1)^{-1}$.
- P is computed using "connected Feynman diagrams" or as the solution of a messy PDE (yet we're still in algebra!).



DoPeGDO Footnotes. †1. Each variable has a "weight" $\in \{0, 1, 2\}$, and always $\text{wt } z_i + \text{wt } \zeta_i = 2$.

- †2. Really, "weight-graded finite sets" $A = A_0 \sqcup A_1 \sqcup A_2$.
- †3. Really, a power series in the weight-0 variables^{†9}.
- †4. The weight of Q must be 2, so it decomposes as $Q = Q_{20} + Q_{11}$. The coefficients of Q_{20} are rational numbers while the coefficients of Q_{11} may be weight-0 power series^{†9}.
- †5. Setting $\text{wt } \epsilon = -2$, the weight of P is ≤ 2 (so the powers of the weight-0 variables are not constrained^{†9}).
- †6. There's also an obvious product $\text{mor}(A_1 \rightarrow B_1) \times \text{mor}(A_2 \rightarrow B_2) \rightarrow \text{mor}(A_1 \sqcup A_2 \rightarrow B_1 \sqcup B_2)$.
- †7. That is, if the weight-0 variables are ignored. Otherwise more care is needed yet the conclusion remains.
- †8. $\text{Hom}(U^{\otimes \Sigma} \rightarrow U^{\otimes S}) \rightsquigarrow \text{mor}(\{\eta_i, \beta_i, \tau_i, \alpha_i, \xi_i\}_{i \in \Sigma} \rightarrow \{y_i, b_i, t_i, a_i, x_i\}_{i \in S})$, where $\text{wt}(\eta_i, \xi_i, y_i, x_i) = 1$ and $\text{wt}(\beta_i, \tau_i, \alpha_i; b_i, t_i, a_i) = (2, 2, 0; 0, 0, 2)$.
- †9. For tangle invariants the wt-0 power series are always rational functions in the exponentials of the wt-0 variables (for knots: just one variable), with degrees bounded linearly by the crossing number.