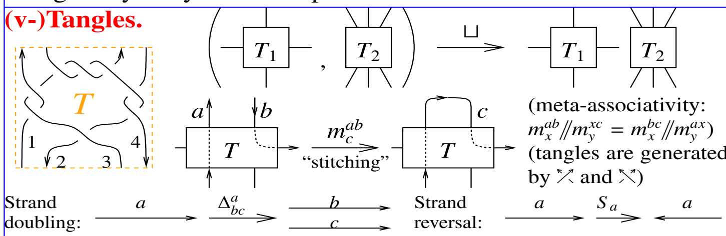
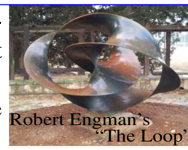
Algebraic Knot Theory

Abstract. This will be a very "light" talk: I will explain why about 13 years ago, in order to have a say on some problems in knot theory, I've set out to find tangle invariants with some nice compositional properties. In other talks in Sydney (ωεβ/talks) I have explained / will explain how such invariants were found - though they are yet to be explored and utilized.

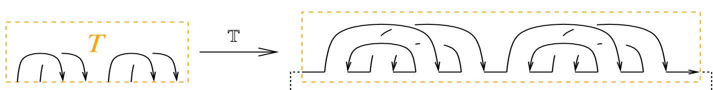
(v-)Tangles.

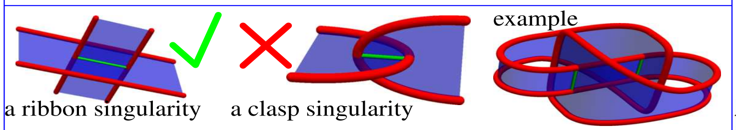


Genus. Every knot is the boundary of an orientable "Seifert Surface" (ωεβ/SS), and the least of their genera is the "genus" of the knot.


Robert Engman's "The Loop"

Claim. The knots of genus ≤ 2 are precisely the images of 4-component tangles via



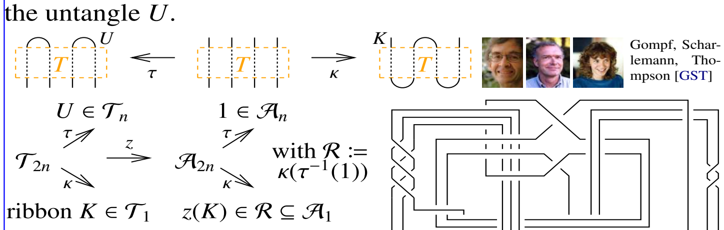


A Bit about Ribbon Knots. A "ribbon knot" is a knot that can be presented as the boundary of a disk that has "ribbon singularities", but no "clasp singularities". A "slice knot" is a knot in $S^3 = \partial B^4$ which is the boundary of a non-singular disk in B^4. Every ribbon knots is clearly slice, yet,

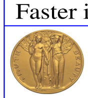
Conjecture. Some slice knots are not ribbon.

Fox-Milnor. The Alexander polynomial of a ribbon knot is always of the form $A(t) = f(t)f(1/t)$. (also for slice)

Theorem. K is ribbon iff it is κT for a tangle T for which τT is the untangle U.



Faster is better, leaner is meaner!



The Gold Standard is set by the "Γ-calculus" Alexander formulas [BNS, BN]. An S-component tangle T has $\Gamma(T) \in R_S \times M_{S\times S}(R_S) = \left\{\begin{array}{c|c}\omega & S\\ \hline S & A\end{array}\right\}$ with $R_S := \mathbb{Z}(\{T_a : a \in S\})$:

$$\left({}_a\!\nearrow\!\!\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\!\!\nearrow_a\right) \to \begin{array}{c|cc} 1 & a & b\\ \hline a & 1 & 1-T_a^{\pm1}\\ b & 0 & T_a^{\pm1}\end{array} \qquad T_1 \sqcup T_2 \to \begin{array}{c|cc} \omega_1\omega_2 & S_1 & S_2\\ \hline S_1 & A_1 & 0\\ S_2 & 0 & A_2\end{array}$$

$$\begin{array}{c|ccc} \omega & a & b & S\\ \hline a & \alpha & \beta & \theta\\ b & \gamma & \delta & \epsilon\\ S & \phi & \psi & \Xi\end{array} \xrightarrow[T_a, T_b \to T_c]{m^{ab}_c} \left(\begin{array}{c|cc} (1-\beta)\omega & c & S\\ \hline c & \gamma + \frac{\alpha\delta}{1-\beta} & \epsilon + \frac{\delta\theta}{1-\beta}\\ S & \phi + \frac{\alpha\psi}{1-\beta} & \Xi + \frac{\psi\theta}{1-\beta}\end{array}\right)$$

For long knots, ω is Alexander, and that's the fastest Alexander algorithm I know! Dunfield: 1000-crossing fast.

Strand Doubling and Reversal.

$$\begin{array}{c|cc} \omega & a & S\\ \hline a & \alpha & \theta\\ S & \phi & \Xi\end{array} \xrightarrow[\substack{\mu := T_a - 1\\ \nu := \alpha - \sigma_a\\ T_a \mapsto T_bT_c}]{q\Delta^a_{bc}} \left(\begin{array}{c|ccc} \omega & b & c & S\\ \hline b & (\sigma_a - \alpha T_a - \nu T_c)/\mu & (T_b - 1)T_c\nu/\mu & (T_b-1)T_c\theta/\mu\\ c & (T_c - 1)\nu/\mu & (\alpha - \sigma_a T_a - \nu T_c)/\mu & (T_c - 1)\theta/\mu\\ S & \phi & \phi & \Xi\end{array}\right)$$

$$dS^a \Big\downarrow T_a \to T_a^{-1}$$

$$\left(\begin{array}{c|cc} \alpha\omega/\sigma_a & a & S\\ \hline a & 1/\alpha & \theta/\alpha\\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha\end{array}\right)$$

Where σ assigns to every $a \in S$ a Laurent monomial σ_a in $\{t_b\}_{b\in S}$ subject to $\sigma\left({}_a\!\nearrow\!\!\!\!\nwarrow_b,\ {}_b\!\nwarrow\!\!\!\!\nearrow_a\right) = (a \to 1, b \to t_a^{\pm1})$, $\sigma(T_1 \sqcup T_2) = \sigma(T_1) \sqcup \sigma(T_2)$, and $\sigma /\!/ m^{ab}_c = (\sigma \setminus \{a,b\}) \cup (c \to \sigma_a\sigma_b)|_{t_a, t_b \to t_c}$.

Vo's Thesis [Vo]. A proof of the Fox-Milnor theorem for ribbon knots using this technology (and more).

Implementation key idea: ωεβ/AlexDemo

$$(\omega, A = (\alpha_{ab})) \leftrightarrow (\omega, \lambda = \textstyle\sum \alpha_{ab} t_a h_b)$$

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Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)a_ :> a, ∞];
   M = Outer[Factor[∂h#1 t#2 λ] &, S, S];
   M = Prepend[M, t# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h# & /@ S, ω]];
   M // MatrixForm];
m[a_, b_ -> c_][Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
   (α β θ; γ δ ϵ; ϕ ψ Ξ) = (∂ta,ha λ  ∂ta,hb λ  ∂ta λ; ∂tb,ha λ  ∂tb,hb λ  ∂tb λ; ∂ha λ  ∂hb λ  λ) /. (t | h)a|b -> 0;
   Γ[(μ = 1 - β) ω, {tc, 1}.(γ + α δ / μ  ϵ + δ θ / μ; ϕ + α ψ / μ  Ξ + ψ θ / μ).{hc, 1}]
     /. {Ta -> Tc, Tb -> Tc} // ΓCollect];
Rp[a_, b_] := Γ[1, {ta, tb}.(1  1 - Ta; 0  Ta).{ha, hb}];
Rm[a_, b_] := Rpab /. Ta -> 1 / Ta;
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Meta-Associativity **Runs.**

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ζ = Γ[ω, {t1, t2, t3, tS}.(α11 α12 α13 θ1; α21 α22 α23 θ2; α31 α32 α33 θ3; ϕ1 ϕ2 ϕ3 Ξ).{h1, h2, h3, hS}];
(ζ // m12→1 // m13→1) == (ζ // m23→2 // m12→1)
True
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R3 ... divide and conquer!

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{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
 Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
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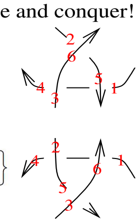
$$\left\{\left(\begin{array}{c|ccc} 1 & h_1 & h_2 & h_3\\ \hline t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} & 1\end{array}\right), \left(\begin{array}{c|ccc} 1 & h_1 & h_2 & h_3\\ \hline t_1 & \frac{T_3}{T_2} & 0 & 0\\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0\\ t_3 & -\frac{-1+T_3}{T_2} & -\frac{-1+T_3}{T_3} & 1\end{array}\right)\right\}$$

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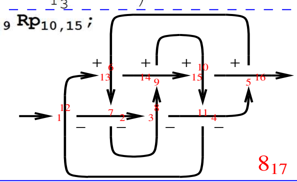
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}];
z
```

$$\left(\begin{array}{c|c} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8\,T_1 + 4\,T_1^2 - T_1^3 & h_1\\ \hline t_1 & 1\end{array}\right)$$



Fact. Γ is better viewed as an invariant of a certain class of 2D knotted objects in $\mathbb{R}^4$ [BND, BN].

Fact. Γ is the "0-loop" part of an invariant that generalizes to "n-loops" (1D tangles only, see further talks and future publications with van der Veen).

Speculation. Stepping stones to categorification?


M. Polyak & T. Ohtsuki @ Heian Shrine, Kyoto

Ask me about geography vs. identity!

References.

[BN] D. Bar-Natan, *Balloons and Hoops and their Universal Finite Type Invariant, BF Theory, and an Ultimate Alexander Invariant,* ωεβ/KBH, arXiv:1308.1721.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I: w-Knots and the Alexander Polynomial,* Alg. and Geom. Top. **16-2** (2016) 1063–1133, arXiv:1405.1956, ωεβ/WKO1.

[BNS] D. Bar-Natan and S. Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* J. of Knot Theory and its Ramifications **22-10** (2013), arXiv:1302.5689.

[GST] R. E. Gompf, M. Scharlemann, and A. Thompson, *Fibered Knots and Potential Counterexamples to the Property 2R and Slice-Ribbon Conjectures,* Geom. and Top. **14** (2010) 2305–2347, arXiv:1103.1601.

[Vo] H. Vo, *Alexander Invariants of Tangles via Expansions,* University of Toronto Ph.D. thesis, ωεβ/Vo.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Macquarie-191016/