A Small-Print Page on ρ_d, $d>1$.

Definition. $\langle f(z_i), h(\zeta_i)\rangle_{\{z_i\}} := f(\partial_{\zeta_i})h\big|_{\zeta_i=0}$, so $\langle p^2x^2, e^{g\pi\xi}\rangle = 2g^2$.

Baby Theorem. There exist (non unique) power series $r^\pm(p_1,p_2,x_1,x_2) = \sum_d \epsilon^d r^\pm_d(p_1,p_2,x_1,x_2) \in \mathbb{Q}[T^{\pm 1}, p_1,p_2,x_1,x_2][\![\epsilon]\!]$ with $\deg r^\pm_d \le 2d+2$ ("docile") such that the power series $Z^b = \sum \rho^b_d \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i,p_j,x_i,x_j)\right), \exp\left(\sum_{\alpha,\beta} g_{\alpha\beta}\pi_\alpha\xi_\beta\right)\right\rangle_{\{p_\alpha,x_\beta\}}$$

is a bnot invariant. Beyond the once-and-for-all computation of $g_{\alpha\beta}$ (a matrix inversion), Z^b is computable in $O(n^d)$ operations in the ring $\mathbb{Q}[T^{\pm 1}]$.

(Bnots are knot diagrams modulo the braid-like Reidemeister moves, but not the cyclic ones).

Theorem. There also exist docile power series $\gamma^\varphi(\bar p,\bar x) = \sum_d \epsilon^d \gamma^\varphi_d \in \mathbb{Q}[T^{\pm 1}, \bar p, \bar x][\![\epsilon]\!]$ such that the power series $Z = \sum \rho_d \epsilon^d :=$

$$\left\langle \exp\left(\sum_c r^s(p_i,p_j,x_i,x_j) + \sum_k \gamma^{\varphi_k}(\bar p_k,\bar x_k)\right),\right.$$
$$\left.\exp\left(\sum_{\alpha,\beta} g_{\alpha\beta}(\pi_\alpha+\bar\pi_\alpha)(\xi_\beta+\bar\xi_\beta) + \sum_\alpha \pi_\alpha\bar\xi_\alpha\right)\right\rangle_{\{p_\alpha,\bar p_\alpha,x_\beta,\bar x_\beta\}}$$

is a knot invariant, as easily computable as Z^b.

Implementation. Data, then program (with output using the Conway variable $z=\sqrt{T}-1/\sqrt{T}$), and then a demo. See `Rho.nb` of ωεβ/ap.

```
V@γ1,φ_[k_] = φ (1/2 - p̄k x̄k);  V@γ2,φ_[k_] = -φ^2 p̄k x̄k / 2;
V@γ3,φ_[k_] := -φ^3 p̄k x̄k / 6
```

```
V@r1,s_[i_, j_] :=
 s (-1 + 2 pi xi - 2 pj xi + (-1 + T^s) pi pj xi^2 + (1 - T^s) pj^2 xi^2 - 2 pi pj xi xj + 2 pj^2 xi xj) / 2
```

```
V@r2,1[i_, j_] :=
 (-6 pi xi + 6 pj xi - 3 (-1 + 3 T) pi pj xi^2 + 3 (-1 + 3 T) pj^2 xi^2 + 4 (-1 + T) pi^2 pj xi^3 -
   2 (-1 + T) (5 + T) pi pj^2 xi^3 + 2 (-1 + T) (3 + T) pj^3 xi^3 + 18 pi pj xi xj -
   18 pj^2 xi xj - 6 pi^2 pj xi^2 xj + 6 (2 + T) pi pj^2 xi^2 xj - 6 (1 + T) pj^3 xi^2 xj -
   6 pi pj^2 xi xj^2 + 6 pj^3 xi xj^2) / 12
```

```
V@r2,-1[i_, j_] :=
 (-6 T^2 pi xi + 6 T^2 pj xi + 3 (-3 + T) T pi pj xi^2 - 3 (-3 + T) T pj^2 xi^2 -
   4 (-1 + T) T pi^2 pj xi^3 + 2 (-1 + T) (1 + 5 T) pi pj^2 xi^3 - 2 (-1 + T) (1 + 3 T) pj^3 xi^3 +
   18 T^2 pi pj xi xj - 18 T^2 pj^2 xi xj - 6 T^2 pi^2 pj xi^2 xj + 6 T (1 + 2 T) pi pj^2 xi^2 xj -
   6 T (1 + T) pj^3 xi^2 xj - 6 T^2 pi pj^2 xi xj^2 + 6 T^2 pj^3 xi xj^2) / (12 T^2)
```

```
V@r3,1[i_, j_] :=
 (4 pi xi - 4 pj xi + 2 (5 + 7 T) pi pj xi^2 - 2 (5 + 7 T) pj^2 xi^2 - 4 (-5 + 6 T) pi^2 pj xi^3 +
   4 (-16 + 17 T + 2 T^2) pi pj^2 xi^3 - 4 (-11 + 11 T + 2 T^2) pj^3 xi^3 + 3 (-1 + T) pi^3 pj xi^4 -
   3 (-1 + T) (4 + 3 T) pi^2 pj^2 xi^4 + (-1 + T) (13 + 22 T + T^2) pi pj^3 xi^4 -
   (-1 + T) (4 + 13 T + T^2) pj^4 xi^4 - 28 pi pj xi xj + 28 pj^2 xi xj + 36 pi^2 pj xi^2 xj -
   12 (9 + 2 T) pi pj^2 xi^2 xj + 24 (3 + T) pj^3 xi^2 xj - 4 pi^3 pj xi^3 xj + 28 T pi^2 pj^2 xi^3 xj -
   4 (-6 + 17 T + T^2) pi pj^3 xi^3 xj + 4 (-5 + 10 T + T^2) pj^4 xi^3 xj + 24 pi pj^2 xi xj^2 -
   24 pj^3 xi xj^2 - 24 pi^2 pj^2 xi^2 xj^2 + 6 (10 + T) pi pj^3 xi^2 xj^2 - 6 (6 + T) pj^4 xi^2 xj^2 -
   4 pi pj^3 xi xj^3 + 4 pj^4 xi xj^3) / 24
```

```
V@r3,-1[i_, j_] :=
 (-4 T^3 pi xi + 4 T^3 pj xi - 2 T^2 (7 + 5 T) pi pj xi^2 + 2 T^2 (7 + 5 T) pj^2 xi^2 -
   4 T^2 (-6 + 5 T) pi^2 pj xi^3 + 4 T (-2 - 17 T + 16 T^2) pi pj^2 xi^3 -
   4 T (-2 - 11 T + 11 T^2) pj^3 xi^3 + 3 (-1 + T) T^2 pi^3 pj xi^4 - 3 (-1 + T) T (3 + 4 T) pi^2 pj^2 xi^4 +
   (-1 + T) (1 + 22 T + 13 T^2) pi pj^3 xi^4 - (-1 + T) (1 + 13 T + 4 T^2) pj^4 xi^4 +
   28 T^3 pi pj xi xj - 28 T^3 pj^2 xi xj - 36 T^3 pi^2 pj xi^2 xj + 12 T^2 (2 + 9 T) pi pj^2 xi^2 xj -
   24 T^2 (1 + 3 T) pj^3 xi^2 xj + 4 T^3 pi^3 pj xi^3 xj - 28 T^2 pi^2 pj^2 xi^3 xj -
   4 T (-1 - 17 T + 6 T^2) pi pj^3 xi^3 xj + 4 T (-1 - 10 T + 5 T^2) pj^4 xi^3 xj -
   24 T^3 pi pj^2 xi xj^2 + 24 T^3 pj^3 xi xj^2 + 24 T^3 pi^2 pj^2 xi^2 xj^2 - 6 T^2 (1 + 10 T) pi pj^3 xi^2 xj^2 +
   6 T^2 (1 + 6 T) pj^4 xi^2 xj^2 + 4 T^3 pi pj^3 xi xj^3 - 4 T^3 pj^4 xi xj^3) / (24 T^3)
```

```
{p*, x*, p̄*, x̄*} = {π, ξ, π̄, ξ̄};  (z_i__)* := (z*)i;
Zip{}[ℰ_] := ℰ;
Zip{z_,zs___}[ℰ_] :=
 (Collect[ℰ // Zip{zs}, z] /. f_. z^d_. :> (D[f, {z*, d}])) /. z* → 0
```

```
gPair[fs_, w_] :=
 gPair[fs, w] =
  Collect[ZipJoin@@Table[{pα,p̄α,xα,x̄α},{α,w}][
    (Times @@ (V /@ fs))
     Exp[Sum[gα,β (πα + π̄α) (ξβ + ξ̄β), {α, w}, {β, w}] - Sum[ξ̄α πα, {α, w}]]],
   g__, Factor]
```

```
T2z[p_] := Module[{q = Expand[p], n, c},
  If[q === 0, 0, c = Coefficient[q, T, n = Exponent[q, T]];
   c z^(2 n) + T2z[q - c (T^(1/2) - T^(-1/2))^(2 n)]]];
```

```
Zd_[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, d1, Z1, Z2, Z3},
  {Cs, φ} = Rot[K]; n = Length[Cs]; A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :> (A[[{i, j}, {i + 1, j + 1}]] += ( -T^s  T^s - 1 ;  0  -1 ))];
  {Δ, G} = Factor@{T^((-Total[φ] - Total[Cs[[All,1]]])/2) Det@A, Inverse@A};
  Z1 =
   Exp[Total[Cases[Cs, {s_, i_, j_} :> Sum[ϵ^d1 rd1,s[i, j], {d1, d}]]] +
     Sum[ϵ^d1 γd1,φ[[k]][k], {k, 2 n}, {d1, d}] /. γ_,0[_] → 0];
  Z2 = Expand[F[{}, {}] × Normal@Series[Z1, {ϵ, 0, d}]] //.
    F[fs_, {es___}] × (f : (r | γ)ps__[is__])^p_. :>
     F[Join[fs, Table[f, p]], DeleteDuplicates@{es, is}];
  Z3 = Expand[Z2 /. F[fs_, es_] :> Expand[gPair[
        Replace[fs, Thread[es → Range@Length@es], {2}], Length@es
        ] /. gα_,β_ :> G[[es[[α]], es[[β]]]] ] ];
  Collect[{Δ, Z3 /. ϵ^p_. → p! Δ^(2 p) ϵ^p}, ϵ, T2z] ];
```

```
Z2[GST48] (* takes a few minutes *)
```

$\{1-4z^2-61z^4-207z^6-296z^8-210z^{10}-77z^{12}-14z^{14}-z^{16},$
$1+(38z^2+255z^4+1696z^6+16\,281z^8+86\,952z^{10}+259\,994z^{12}+487\,372z^{14}+615\,066z^{16}+543\,148z^{18}+341\,714z^{20}+153\,722z^{22}+48\,983z^{24}+10\,776z^{26}+1554z^{28}+132z^{30}+5z^{32})\,\epsilon+$
$(-8-484z^2+9709z^4+165\,952z^6+1\,590\,491z^8+16\,256\,508z^{10}+115\,341\,797z^{12}+432\,685\,748z^{14}+395\,838\,354z^{16}-4\,017\,557\,792z^{18}-23\,300\,064\,167z^{20}-70\,082\,264\,972z^{22}-142\,572\,271\,191z^{24}-209\,475\,503\,700z^{26}-221\,616\,295\,209z^{28}-151\,502\,648\,428z^{30}-23\,700\,199\,243z^{32}+99\,462\,146\,328z^{34}+164\,920\,463\,074z^{36}+162\,550\,825\,432z^{38}+119\,164\,552\,296z^{40}+69\,153\,062\,608z^{42}+32\,547\,596\,611z^{44}+12\,541\,195\,448z^{46}+3\,961\,384\,155z^{48}+1\,021\,219\,696z^{50}+212\,773\,106z^{52}+35\,264\,208z^{54}+4\,537\,548z^{56}+436\,600z^{58}+29\,536z^{60}+1252z^{62}+25z^{64})\,\epsilon^2\}$

```
TableForm[Table[Join[{K[[1]]K[[2]]}, Z3[K]], {K, AllKnots[{3, 6}]}], TableAlignments → Center] (* takes a few minutes *)
```

3_1	$1+z^2$	$1+(2z^2+z^4)\epsilon+(2-4z^2+3z^4+4z^6+z^8)\epsilon^2+(-12+74z^2-27z^4-20z^6+8z^8+6z^{10}+z^{12})\epsilon^3$
4_1	$1-z^2$	$1+(-2+2z^4)\epsilon^2$
5_1	$1+3z^2+z^4$	$1+(10z^2+21z^4+12z^6+2z^8)\epsilon+(6-28z^2+33z^4+364z^6+655z^8+536z^{10}+227z^{12}+48z^{14}+4z^{16})\epsilon^2+(-60+970z^2+645z^4-3380z^6-3280z^8+7470z^{10}+19\,475z^{12}+20\,536z^{14}+12\,564z^{16}+4774z^{18}+1109z^{20}+144z^{22}+8z^{24})\epsilon^3$
5_2	$1+2z^2$	$1+(6z^2+5z^4)\epsilon+(4-20z^2+43z^4+64z^6+26z^8)\epsilon^2+(-36+498z^2-883z^4+100z^6+816z^8+556z^{10}+146z^{12})\epsilon^3$
6_1	$1-2z^2$	$1+(-2z^2+z^4)\epsilon+(-4+4z^2+25z^4-8z^6+2z^8)\epsilon^2+(12+154z^2-223z^4-608z^6+100z^8-52z^{10}+10z^{12})\epsilon^3$
6_2	$1-z^2-z^4$	$1+(-2z^2-3z^4+2z^6+z^8)\epsilon+(-2-4z^2+29z^4+28z^6+42z^8-8z^{10}-2z^{12}+4z^{14}+z^{16})\epsilon^2+(12+166z^2+155z^4-194z^6-2453z^8-1622z^{10}-1967z^{12}-258z^{14}+49z^{16}-30z^{18}+z^{20}+6z^{22}+z^{24})\epsilon^3$
6_3	$1+z^2+z^4$	$1+(2+8z^2-16z^6-24z^8-16z^{10}-2z^{12})\epsilon^2$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Oaxaca-2210/