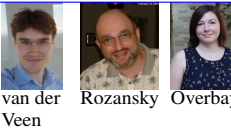


Cars, Interchanges, Traffic Counters, and some Pretty Darned Good Knot Invariants

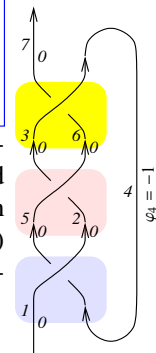
More at œβ/APAI



Abstract. Reporting on joint work with Roland van der Veen, I'll tell you some stories about ρ_1 , an easy to define, strong, fast to compute, homomorphic, and well-connected knot invariant. ρ_1 was first studied by Rozansky and Overbay [Ro1, Ro2, Ro3, Ov], it has far-reaching generalizations, it is dominated by the coloured Jones polynomial, and I wish I understood it. **Common misconception.** "Dominated" \Rightarrow "lesser".



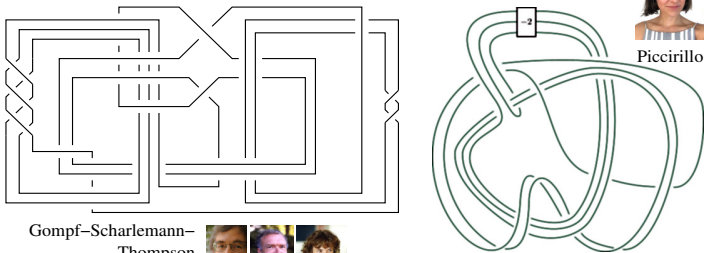
Jones:
Formulas stay;
interpretations change with time.



We seek strong, fast, homomorphic knot and tangle invariants.

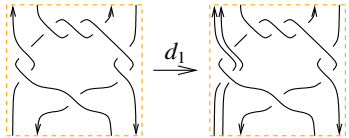
Strong. Having a small "kernel".

Fast. Computable even for large knots (best: poly time).

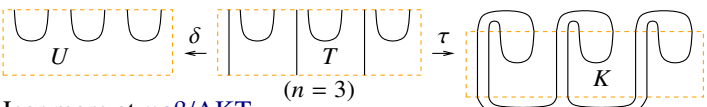


Gompf-Scharlemann-Thompson

Homomorphic. Extends to tangles and behaves under tangle operations; especially gluings and doublings:



Why care for "Homomorphic"? **Theorem.** A knot K is ribbon iff there exists a $2n$ -component tangle T with skeleton as below such that $\tau(T) = K$ and where $\delta(T) = U$ is the untangle:



Hear more at œβ/AKT.

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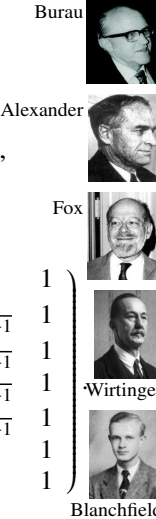
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Formulas. Draw an n -crossing knot K as on the right: all crossings face up, and the edges are marked with a running index $k \in \{1, \dots, 2n + 1\}$ and with rotation numbers φ_k . Let A be the $(2n + 1) \times (2n + 1)$ matrix constructed by starting with the identity matrix I , and adding a 2×2 block for each crossing:

$$c : \begin{matrix} s = +1 & s = -1 \\ j+1 \uparrow & i+1 \uparrow \\ i & j \end{matrix} \begin{matrix} i+1 \downarrow & j+1 \downarrow \\ j & i \end{matrix} \longrightarrow \begin{matrix} A & \text{col } i+1 & \text{col } j+1 \\ \text{row } i & -T^s & T^s - 1 \\ \text{row } j & 0 & -1 \end{matrix}$$

Let $G = (g_{\alpha\beta}) = A^{-1}$. For the trefoil example, it is:

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Note. The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2} \det(A), \quad \text{with } \varphi = \sum_k \varphi_k, \quad w = \sum_c s.$$

Classical Topologists: This is boring. Yawn.

Formulas, continued. Finally, set

$$R_1(c) := s(g_{ji}(g_{j+1,j} + g_{j,j+1} - g_{ij}) - g_{ii}(g_{j,j+1} - 1) - 1/2)$$

$$\rho_1 := \Delta^2 \left(\sum_c R_1(c) - \sum_k \varphi_k (g_{kk} - 1/2) \right).$$

In our example $\rho_1 = -T^2 + 2T - 2 + 2T^{-1} - T^{-2}$.

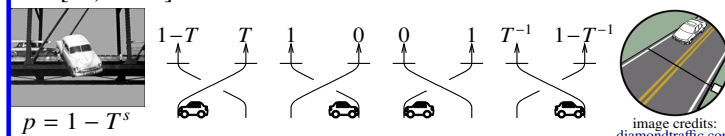
Theorem. ρ_1 is a knot invariant.

Proof: later.

Classical Topologists: Whiskey Tango Foxtrot?

Cars, Interchanges, and Traffic Counters.

Cars always drive forward. When a car crosses over a bridge it goes through with (algebraic) probability $T^s \sim 1$, but falls off with probability $1 - T^s \sim 0^*$. See also [Jo, LTW].



* In algebra $x \sim 0$ if for every y in the ideal generated by x , $1 - y$ is invertible.