Abstract. My talk had two parts:

• In the first part I described the (tentative and speculative) “Projectivization Paradigm”, which says, roughly speaking, that everything graded and interesting is the associated graded of something plain (“ungraded”, “global”) and even more interesting. The paradigm is absolutely general, encompassing practically every algebraic structure that might exist, and there is a diverse base of interesting examples and candidates for future examples.

• In the second part I described my latest example of an instance of the Projectivization Paradigm. I showed that the projectivization of “the circuit algebra of welded tangles” describes a good part (and maybe, in the future, all) of the recent work by Alekseev and Torossian on Drinfel’d associators and the Kashiwara-Vergne conjecture. This is cool: it leads to a nice conceptual construction of tree-level associators which might even be brought to a closed form, and it seems like a step towards a better understanding of quantum universal enveloping algebras and the work of Etingof and Kazhdan.

The work is very new. I’m quite confident of the overall picture but the details are subject to change.

To a very large extent my talk followed the two-page handout attached as the last two pages of this document.

1. The Projectivization Speculative Paradigm

I started by reminding the conference about the “Categorification Speculative Paradigm”, which says, in very rough terms, that all of mathematics, or at least all of integer-coefficient mathematics, is the “Euler shadow” of vector-space, homological, mathematics. This, of course, is merely a speculative paradigm. One cannot expect it to be literally true, yet it is an excellent guiding principle for research. A lot of interesting mathematics arises as one tries to explore the extent to which this speculative paradigm holds true.

In a similar manner I proposed the “Projectivization Tentative Speculative Paradigm”, which says, in very rough terms, that all of graded mathematics is the projectivization of “plain”, “ungraded” or “global” mathematics: all graded algebraic structures are the projectivizations of global ones, and all graded equations are the equations for “homomorphic expansions”, or for “automorphisms” of homomorphic expansions.

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1“Tentative” because I’m not even sure if the name “projectivization” (meant to be catchy and convey a “graded” feeling) is appropriate.
I then proceeded to explain most of the terms appearing in the above paragraph. For a start, I gave a few examples of “graded equations” (these are the entities the projectivization paradigm is supposed to explain):

- The exponential equation $e(x + y) = e(x)e(y)$ [BN4].
- The pentagon and hexagon equations for Drinfel’d associators [Dr2, Dr3, BN1, BN2].
- The equations defining a quantized universal enveloping algebra in the sense of Drinfel’d [Dr1] and Etingof-Kazhdan [EK]. For the long term, these are the equations I care about the most, and my dream is to eventually incorporate them to within the projectivization paradigm.
- The equations appearing in the Alekseev-Torossian work [AT] on Drinfel’d associators and the Kashiwara-Vergne Conjecture [KV]. These equations are the main concern of the second part of this talk. One wonderful feature of these equations is that (in suitable quotients) they have explicit solutions, that will likely lead to explicit formulas for tree-level associators.

I then moved on to explain what is “the projectivization of an algebraic structure”. For this purpose, an “algebraic structure” $\mathcal{O}$ is practically anything that is made of “spaces” and “operations”. Allowing for formal linear combinations and extending all operations in a multi-linear manner, we can always define an “augmentation ideal” $I$ along with its powers $I^n$, and then we can set

$$\text{proj } \mathcal{O} := \bigoplus_{n \geq 0} I^n/I^{n+1}.$$ 

One can see that proj $\mathcal{O}$ is endowed with the same operations as $\mathcal{O}$, though they need not satisfy the same “axioms” that the operations of $\mathcal{O}$ may satisfy. We noted that if $\mathcal{O}$ is an appropriate space of knotted objects, then proj $\mathcal{O}$ is the corresponding space of “chord diagrams”.

Some warm up examples followed. We noted that the projectivization of a group is a graded associative algebra, and that the projectivization of a quandle is a graded Lie algebra.

I then moved on to discuss the central notion in the statement of the projectivization paradigm — the notion of an “expansion” [Li], and more importantly, of a “homomorphic expansion” — a “homomorphism” $Z: \mathcal{O} \rightarrow \text{proj } \mathcal{O}$ which “covers” the identity map on proj $\mathcal{O}$. When $\mathcal{O}$ is “finitely presented”, finding an expansion involves finding values for $Z(g_i)$ (where the $g_i$’s are the generators of $\mathcal{O}$), where these values must satisfy the equations corresponding to the “defining relations” of $\mathcal{O}$. Hence as promised$^2$ in the statement of the projectivization paradigm, finding a homomorphic expansion is a matter of solving equations in a graded space, proj $\mathcal{O}$.

A pretty example involves “knotted trivalent graphs” [BN3]. Here the relevant algebraic structure $\mathcal{O} = \text{KTG}$ has a “space” for each trivalent graph — the space of “knottings” of that graph, and the operations are “delete”, “unzip” and “connected sum”. With these operations KTG is finitely generated, with the most interesting generator being the unknotted tetrahedron $T$. The interesting relations that $T$ satisfies turn out to be (after appropriate language changes) the pentagon and the hexagons, and therefore it turns out that the

$^2$The other source of graded equations, “automorphisms of homomorphic expansions”, was not discussed in my talk.
equations for a “homomorphic expansion” for KTG are equivalent\(^3\) to the equations for an associator.

I then explained how homomorphic expansions may be used — they convert certain kinds of “global” problems into problems that can be addressed “degree by degree”. In the case of knotted trivalent graphs we arrive at what one may call “Algebraic Knot Theory” [BN3]. Certain knot theoretic properties, such as the knot genus and the property of being a ribbon, are “definable” using “delete”, “unzip” and “connected sum”, and hence they are in principle susceptible to study using homomorphic expansions.

2. Welded Knots and Alekseev-Torossian

Due to time constraints, the second half of my talk had to be sketchy. Following a talk Lou Kauffman gave in 2001, I recalled virtual knots [Ka], welded knots [FRR], and the relationship between welded knots and tori in \(\mathbb{R}^4\) [Sa].

Welded knots form a “circuit algebra”, and as a circuit algebra, their projectivization turns out to contain all the spaces (most notably \(\text{tder}_n\), \(\text{sder}_n\) and \(\text{tr}_n\)) considered by Alekseev and Torossian [AT]. As a circuit algebra, the related space of “welded trivalent graphs” is generated by the “Y-vertex” and by crossings. Calling the images of these generators via a homomorphic expansion \(F\) and \(R\), we find that \(F\) and \(R\) need to satisfy some equations — precisely the equations studied by Alekseev and Torossian. Finally, as welded trivalent graphs contain a quotient of knotted trivalent graphs, the Alekseev-Torossian theory contains a quotient of the Drinfel’d theory, which turns out to be the theory of tree-level associators.

3. Propaganda

“God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)

References


\(^3\)Well, at least if one ignores the fine print. The precise statement is a bit longer but follows the same spirit.


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An Algebraic Structure

The Categorification Speculative Paradigm.
- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm.
- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples
- The pentagon and hexagons in $\mathcal{A}(1,4)$.
- The equations defining a QEA, the work of Etingof and Kazhdan.

The Alekseev-Torossian equations in $\mathcal{U}(sder_n)$ and $\mathcal{U}(tder_n)$.

$F \in \mathcal{U}(tder_2)$: 
\[
F^{-1} e(x + y) F = e(x)e(y) \iff F \in \text{Sol}_0
\]

$\Phi = \Phi_F := (F^{23})^{-1}(F^{12})^{-1} F^{2} F^{1} \in \mathcal{U}(sder_3)$

$\Phi^{1,2,3} \Phi^{2,3,4} \Phi^{3,4,1} \Phi^{1,4,2,3} = \Phi^{1,2,3,4} \Phi^{1,2,3,4} \Phi^{1,2,3,4}$

"the pancake" 

$1 = e(\tau) \in sder_2$ satisfies $4\tau$ and $r = (y, 0) \in tder_2$ satisfies $6\tau$

$R := e(r)$ satisfies Yang-Baxter: 

$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}$

also $R^{12} = R^{13} R^{23}$ and $F^{23} R^{23} (F^{23})^{-1} = R^{23} R^{13}$

$\tau(F) := RT^{23} e(-t)$ is an involution, $\Phi_{\tau(F)} = (\Phi^{23})^{-1}$

$\text{Sol}_0 := \{ F : \tau(F) = F \}$ is non-empty; for $F \in \text{Sol}_0$, 

$e(t^{13} + t^{23}) = \Phi^{121} e(t^{13}) (\Phi^{231})^{-1} e(t^{23}) \Phi^{231}$

and $e(t^{12} + t^{13}) = (\Phi^{123})^{-1} e(t^{13}) \Phi^{123} e(t^{13}) \Phi^{123}$

This is just a part of the Alekseev-Torossian work!

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QEA equations; we may be on the right track!

Knot Theory Anchors.
- $(\mathcal{O}(\mathcal{D})_n)^*$ is "type n invariants".
- $(\mathcal{O}(\mathcal{D})_n)^+$ is "weight systems".
- proj $\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".

Warmup Examples.
- The projectivization of a group is a graded associative algebra.
- A quandle: a set $Q$ with a binary operation $\cdot$ such that $1 \cdot x = x, x \cdot 1 = x \land x = x, (x \cdot y) \land z = (x \land y) \land z$.

An Expansion is $Z: \mathcal{O} \leftrightarrow \text{proj} \mathcal{O}$ s.t. $Z(I^n) \subset \text{proj}(\mathcal{O})_n$ and $Z(I^n_{\mathcal{O}}) = I d_{\text{proj} \mathcal{O}}$ (A "universal finite type invariant"). In practice, it is hard to determine $\text{proj} \mathcal{O}$, but easy to guess a surjection $\rho : A \rightarrow \text{proj} \mathcal{O}$. So find $Z': \mathcal{O} \rightarrow A$ with $Z'(I^n) \subset A_{\mathcal{O}}$ and $Z'(I^n_{\mathcal{O}}) \subset \rho_{\mathcal{O}} \circ Id_{\mathcal{A}_n}$.

Homomorphic Expansions are expansions that intertwine the algebraic structure on $\mathcal{O}$ and proj $\mathcal{O}$. They provide finite / combinatorial handles on global problems.
Circuit Algebras

* Have “circuits” with “ends”
* Can be wired arbitrarily.
* May have “relations” – de-Morgan, etc.

“Welded trivalent (framed) tangles” are a circuit algebra:

WT = \langle Y, Y \rangle / R123, R4 (for vertices), F1.

Further operations: delete, unzip.

The “Chord Diagrams” – \( A^c_n \): As we did for quandles, substitute

\[ x \rightarrow x + (x - x) = x + x \]

into the various maps to get relations. Also, switch to a “diagram language”:

\[ x \rightarrow (\text{tangle}) \text{ where } \]

\[ R3 \rightarrow - \]

\[ R4 \rightarrow \]

\[ \text{tangles commutate} \]

The “Jacobi Diagrams” – \( A^w_n \).

Theorem. (95%) \( A^{jw}_n \) is \( A^{cc}_n \) is \( U(tder_n) \).

Partial Dictionary.

\[ R,F \rightarrow \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \]

Further operations: delete, unzip.

The Map \( \alpha \): \( A^{tree}_n \rightarrow A^c_n \).

Theorem. (90%) \( \alpha \) is an injection on \( A^{tree}_n \approx U(tder_n) \).

Furthermore, there is a simple characterization of \( \text{im } \alpha \), so we can tell “an arrowless element” when we see it.

The Main Theorem. (80%/0%) \( F \)'s in Sol_0 are in a bijective correspondence with tree-level associators for ordinary parenthesized tangles (or ordinary knotted trivalent graphs) / with homomorphic expansions for knotted welded trivalent tangles.

Disclaimer: Orientations, rotation numbers, framings, the vertical direction and the cyclic symmetry of the vertex may still make everything uglier. I hope not.