PROJECTIVIZATION, WELDED KNOTS AND ALEKSEEV-TOROSSIAN (SUMMARY OF OBERWOLFACH TALK, GIVEN MAY 9, 2008)

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Abstract. My talk had two parts:

- In the first part I described the (tentative and speculative) "Projectivization Paradigm", which says, roughly speaking, that everything graded and interesting is the associated graded of something plain ("ungraded", "global") and even more interesting. The paradigm is absolutely general, encompassing practically every algebraic structure that might exist, and there is a diverse base of interesting examples and candidates for future examples.
- In the second part I described my latest example of an instance of the Projectivization Paradigm: I showed that the projectivization of "the circuit algebra of welded tangles" describes a good part (and maybe, in the future, all) of the recent work by Alekseev and Torossian on Drinfel'd associators and the Kashiwara-Vergne conjecture. This is cool: it leads to a nice conceptual construction of tree-level associators which might even be brought to a closed form, and it seems like a step towards a better understanding of quantum universal enveloping algebras and the work of Etingof and Kazhdan.

The work is very new. I'm quite confident of the overall picture but the details are subject to change.

To a very large extent my talk followed the two-page handout attached as the last two pages of this document.

1. The Projectivization Speculative Paradigm

I started by reminding the conference about the "Categorification Speculative Paradigm", which says, in very rough terms, that all of mathematics, or at least all of integer-coefficient mathematics, is the "Euler shadow" of vector-space, homological, mathematics. This, of course, is merely a speculative paradigm. One cannot expect it to be literally true, yet it is an excellent guiding principle for research. A lot of interesting mathematics arises as one tries to explore the extent to which this speculative paradigm holds true.

In a similar manner I proposed the "Projectivization Tentative[1] Speculative Paradigm", which says, in very rough terms, that all of graded mathematics is the projectivization of "plain", "ungraded" or "global" mathematics: all graded algebraic structures are the projectivizations of global ones, and all graded equations are the equations for "homomorphic expansions", or for "automorphisms" of homomorphic expansions.

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Key words and phrases. Projectivization, Welded knots, Algebraic structures, Finite type invariants, Chord diagrams, Algebraic knot theory, Circuit algebras, Virtual knots, Kashiwara-Vergne.

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[1] "Tentative" because I'm not even sure if the name "projectivization" (meant to be catchy and convey a "graded" feeling) is appropriate.

I then proceeded to explain most of the terms appearing in the above paragraph. For a start, I gave a few examples of "graded equations" (these are the entities the projectivization paradigm is supposed to explain):

- The exponential equation $e(x+y) = e(x)e(y)$ [BN4].
- The pentagon and hexagon equations for Drinfel'd associators [Dr2, Dr3, BN1, BN2].
- The equations defining a quantized universal enveloping algebra in the sense of Drinfel'd [Dr1] and Etingof-Kazhdan [EK]. For the long term, these are the equations I care about the most, and my dream is to eventually incorporate them to within the projectivization paradigm.
- The equations appearing in the Alekseev-Torossian work [AT] on Drinfel'd associators and the Kashiwara-Vergne Conjecture [KV]. These equations are the main concern of the second part of this talk. One wonderful feature of these equations is that (in suitable quotients) they have explicit solutions, that will likely lead to explicit formulas for tree-level associators.

I then moved on to explain what is "the projectivization of an algebraic structure". For this purpose, an "algebraic structure" $\mathcal{O}$ is practically anything that is made of "spaces" and "operations". Allowing for formal linear combinations and extending all operations in a multi-linear manner, we can always define an "augmentation ideal" I along with its powers I^n, and then we can set

$$\operatorname{proj}\mathcal{O} := \bigoplus_{n\geq 0} I^n/I^{n+1}.$$

One can see that $\operatorname{proj}\mathcal{O}$ is endowed with the same operations as $\mathcal{O}$, though they need not satisfy the same "axioms" that the operations of $\mathcal{O}$ may satisfy. We noted that if $\mathcal{O}$ is an appropriate space of knotted objects, then $\operatorname{proj}\mathcal{O}$ is the corresponding space of "chord diagrams".

Some warm up examples followed. We noted that the projectivization of a group is a graded associative algebra, and that the projectivization of a quandle is a graded Lie algebra.

I then moved on to discuss the central notion in the statement of the projectivization paradigm — the notion of an "expansion" [Li], and more importantly, of a "homomorphic expansion" — a "homomorphism" $Z\colon \mathcal{O} \to \operatorname{proj}\mathcal{O}$ which "covers" the identity map on $\operatorname{proj}\mathcal{O}$. When $\mathcal{O}$ is "finitely presented", finding an expansion involves finding values for $Z(g_i)$ (where the g_i's are the generators of $\mathcal{O}$), where these values must satisfy the equations corresponding to the "defining relations" of $\mathcal{O}$. Hence as promised[2] in the statement of the projectivization paradigm, finding a homomorphic expansion is a matter of solving equations in a graded space, $\operatorname{proj}\mathcal{O}$.

A pretty example involves "knotted trivalent graphs" [BN3]. Here the relevant algebraic structure $\mathcal{O} = \text{KTG}$ has a "space" for each trivalent graph — the space of "knottings" of that graph, and the operations are "delete", "unzip" and "connected sum". With these operations KTG is finitely generated, with the most interesting generator being the unknotted tetrahedron T. The interesting relations that T satisfies turn out to be (after appropriate language changes) the pentagon and the hexagons, and therefore it turns out that the

[2]The other source of graded equations, "automorphisms of homomorphic expansions", was not discussed in my talk.

equations for a "homomorphic expansion" for KTG are equivalent[3] to the equations for an associator.

I then explained how homomorphic expansions may be used — they convert certain kinds of "global" problems into problems that can be addressed "degree by degree". In the case of knotted trivalent graphs we arrive at what one may call "Algebraic Knot Theory" [BN3]. Certain knot theoretic properties, such as the knot genus and the property of being a ribbon, are "definable" using "delete", "unzip" and "connected sum", and hence they are in principle susceptible to study using homomorphic expansions.

2. Welded Knots and Alekseev-Torossian

Due to time constraints, the second half of my talk had to be sketchy. Following a talk Lou Kauffman gave in 2001, I recalled virtual knots [Ka], welded knots [FRR], and the relationship between welded knots and tori in $\mathbb{R}^4$ [Sa].

Welded knots form a "circuit algebra", and as a circuit algebra, their projectivization turns out to contain all the spaces (most notably tder_n, sder_n and tr_n) considered by Alekseev and Torossian [AT]. As a circuit algebra, the related space of "welded trivalent graphs" is generated by the "Y-vertex" and by crossings. Calling the images of these generators via a homomorphic expansion F and R, we find that F and R need to satisfy some equations — precisely the equations studied by Alekseev and Torossian. Finally, as welded trivalent graphs contain a quotient of knotted trivalent graphs, the Alekseev-Torossian theory contains a quotient of the Drinfel'd theory, which turns out to be the theory of tree-level associators.

3. Propaganda

"God created the knots,
all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)

The Knot Atlas
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[3] Well, at least if one ignores the fine print. The precise statement is a bit longer but follows the same spirit.

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Projectivization, Welded Knots and Alekseev-Torossian

The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm.

Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples

- $e(x+y)=e(x)e(y)$ in $\mathbb{Q}[[x,y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.
- The Alekseev-Torossian equations in $\mathcal{U}(\mathrm{sder}_n)$ and $\mathcal{U}(\mathrm{tder}_n)$.

sder $\leftrightarrow$ tree-level $\mathcal{A}$
tder $\leftrightarrow$ more

$$F\in\mathcal{U}(\mathrm{tder}_2);\quad F^{-1}e(x+y)F=e(x)e(y)\quad\Longleftrightarrow\quad F\in\mathrm{Sol}_0$$

$$\Phi=\Phi_F:=(F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23}\in\mathcal{U}(\mathrm{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4}=\Phi^{12,3,4}\Phi^{1,2,34}\qquad\text{"the pentagon"}$$

$$t=\frac{1}{2}(y,x)\in\mathrm{sder}_2\text{ satisfies }4T\quad\text{and}\quad r=(y,0)\in\mathrm{tder}_2\text{ satisfies }6T$$

$$R:=e(r)\text{ satisfies Yang-Baxter: }R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$$

$$\text{also }R^{12,3}=R^{13}R^{23}\text{ and }F^{23}R^{1,23}(F^{23})^{-1}=R^{12}R^{13}$$

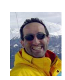
$$\tau(F):=RF^{21}e(-t)\text{ is an involution},\Phi_{\tau(F)}=(\Phi_F^{321})^{-1}$$

$$\mathrm{Sol}_0^\tau:=\{F\colon\tau(F)=F\}\text{ is non-empty; for }F\in\mathrm{Sol}_0^\tau,$$

$$e(t^{13}+t^{23})=\Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and}\quad e(t^{12}+t^{13})=(\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$


Alekseev


Torossian

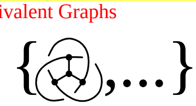
This is just a part of the Alekseev-Torossian work!

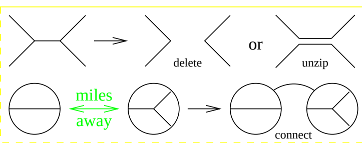
So What?

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

Knotted Trivalent Graphs

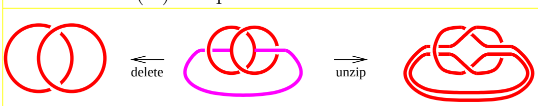
$\mathcal{O}(\mathbin{\triangle}) = $





Theorem. KTG is generated by the unknotted $\triangle$ and the Möbius band, with identifiable relations between them.
Theorem. $Z(\triangle)$ is equivalent to an associator Φ.

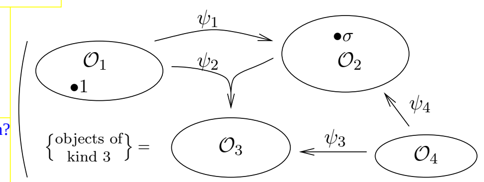
Algebraic Knot Theory



Theorem. $\{\text{ribbon knots}\}\sim\{u\gamma\colon\gamma\in\mathcal{O}(\infty),\ d\gamma=\bigcirc\bigcirc\}$.
Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.

"An Algebraic Structure"

$\mathcal{O}=$



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining proj $\mathcal{O}$.

The augmentation "ideal":

$$I=I_{\mathcal{O}}:=\left\{\begin{array}{l}\text{formal differences of ob-}\\\text{jects "of the same kind"}\end{array}\right\}.$$

Then $I^n:=\left\{\begin{array}{l}\text{all outputs of algebraic}\\\text{expressions at least } n \text{ of}\\\text{whose inputs are in } I\end{array}\right\}$, and

$$\operatorname{proj}\mathcal{O}:=\bigoplus_{n\geq 0}I^n/I^{n+1}.$$

- Has same kinds and operations, but different objects and axioms.

Knot Theory Anchors.

- $(\mathcal{O}/I^{n+1})^\star$ is "type n invariants".
- $(I^n/I^{n+1})^\star$ is "weight systems".
- proj $\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".


Vassiliev


Goussarov

Warmup Examples.

- The projectivization of a group is a graded associative algebra.
- A quandle: a set Q with a binary op $\wedge$ s.t.

$$1\wedge x=1,\quad x\wedge 1=x\wedge x=x,\qquad\text{(appetizers)}$$
$$(x\wedge y)\wedge z=(x\wedge z)\wedge(y\wedge z).\qquad\text{(main)}$$

$L:=\operatorname{proj}Q$ is a graded Lie algebra: set $\bar v:=(v-1)$ (these generate I!), feed $1+\bar x$, $1+\bar y$, $1+\bar z$ in (main), and collect the surviving terms of lowest degree:

$$(\bar x\wedge\bar y)\wedge\bar z=(\bar x\wedge\bar z)\wedge\bar y+\bar x\wedge(\bar y\wedge\bar z).$$

An Expansion

is $Z\colon\mathcal{O}\to\operatorname{proj}\mathcal{O}$ s.t. $Z(I^n)\subset(\operatorname{proj}\mathcal{O})_{\geq n}$ and $Z_{I^n/I^{n+1}}=Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). In practice, it is hard to determine proj $\mathcal{O}$, but easy to guess a surjection $\rho:\mathcal{A}\to\operatorname{proj}\mathcal{O}$. So find $Z'\colon\mathcal{O}\to\mathcal{A}$ with $Z'(I^n)\subset\mathcal{A}_{\geq n}$ and $Z'_{I^n/I^{n+1}}\circ\rho_n=Id_{\mathcal{A}_n}$:

$$\begin{tikzcd}\mathcal{O}\arrow[r,"Z'"]\arrow[dr,"Z"'] & \mathcal{A}\arrow[d,"\rho"] & \mathcal{A}_n\arrow[l,leftrightarrow] \\ & \operatorname{proj}\mathcal{O}\arrow[r,leftrightarrow] & (\operatorname{proj}\mathcal{O})_n\arrow[u,"Z'_{I^n/I^{n+1}}"']\end{tikzcd}$$

Can you make this diagram less confusing?

Homomorphic Expansions

are expansions that intertwine the algebraic structure on $\mathcal{O}$ and proj $\mathcal{O}$. They provide finite / combinatorial handles on global problems.


X-S. Lin

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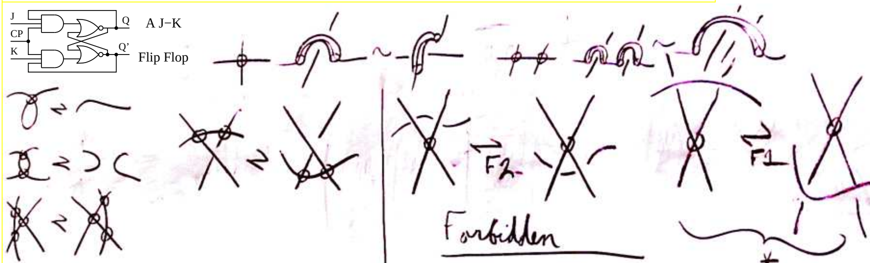
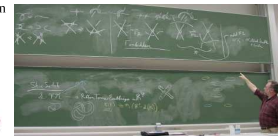
Projectivization, Welded Knots and Alekseev-Torossian



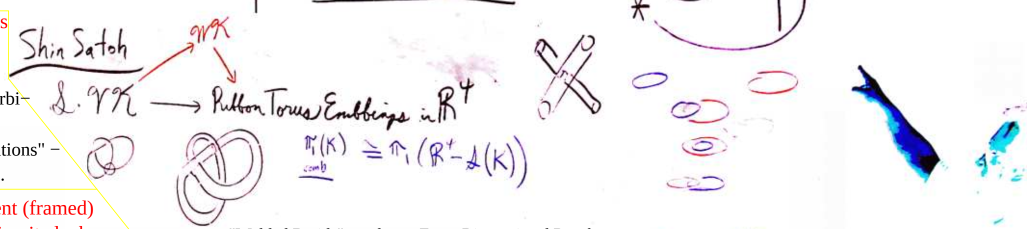
Kauffman in Siegen



Circuit Algebras

* Have "circuits" with "ends"
* Can be wired arbitrarily.
* May have "relations" - de-Morgan, etc.

"Welded trivalent (framed) tangles" are a circuit algebra:



"Welded Braids" are due to Fenn, Rimanyi and Rourke

$\mathcal{WT} = \langle \ \rangle /$ R123, R4 (for vertices), F1.
Further operations: delete, unzip.

The "Chord Diagrams" — $\mathcal{A}_n^{wt}$. As we did for quandles, substitute into the various moves, to get relations. Also switch to "arrow diagram language". Get:

(tails commute)

R3 ⟼ (4T really 6T)

R4 ⟼ $= 0$ (vertex invariance)

The "Jacobi Diagrams" — $\mathcal{A}_n^{cc}$.

Theorem. (95%) $\mathcal{A}_n^{wt}$ is $\mathcal{A}_n^{cc}$ is $\mathcal{U}(\mathrm{tder}_n)$.

(Pretend wheels die: $= 0$)

Here $\mathcal{A}_n^{cc}$ is $\{\ \}_n$ / rels

trivalent directed trees with only 2-in 1-out vertices

In tensorland, this is "Co-Commutative Lie-bialg"

Rels: tails commute

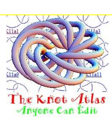
Heads satisfy the only possible STU: + also IHX and vertex invariance

The Map $\alpha\colon \mathcal{A}_n^{tree} \to \mathcal{A}_n^{cc}$:

Theorem. (90%) α is an injection on $\mathcal{A}_n^{tree} \cong \mathcal{U}(\mathrm{sder}_n)$. Furthermore, there is a simple charactarization of im α, so we can tell "an arrowless element" when we see it.

"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)



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Partial Dictionary.

$(R, F) \leftrightarrow (\ ,\ )$ $(r, t) \leftrightarrow (\ ,\ )$

$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \leftrightarrow$

$FF^{-1} = I \leftrightarrow$ unzip

$F^{-1}e(x+y)F = e(x)e(y)$

$F^{23}R^{1,23} = R^{12}R^{13}F^{23} \leftrightarrow$

$R^{12,3} = R^{13}R^{23} \leftrightarrow$

$F^{12,3}R^{12,3} = R^{13}R^{23}F^{12,3}$ (unforbidding F1 makes this automatic)

$RF^{21}e(-t) = F \leftrightarrow$

$\Phi = (F^{12,3})^{-1}(F^{1,2})^{-1}F^{2,13}F^{1,23} \leftrightarrow$ = "The tetrahedron"

$\Phi_{ESDU} \leftrightarrow$

The pentagon and the hexagons follow, with a minor twist, from the fact that we have an unzip-behaved invariant of KTG's.

The Main Theorem. (80%/0%) F's in Sol_0^τ are in a bijective correspondance with tree-level associators for ordinary paranthesized tangles (or ordinary knotted trivalent graphs) / with homomorphic expansions for knotted welded trivalent tangles.

Disclaimer. Orientations, rotation numbers, framings, the vertical direction and the cyclic symmetry of the vertex may still make everything uglier. I hope not.