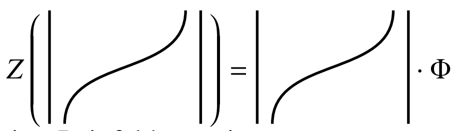
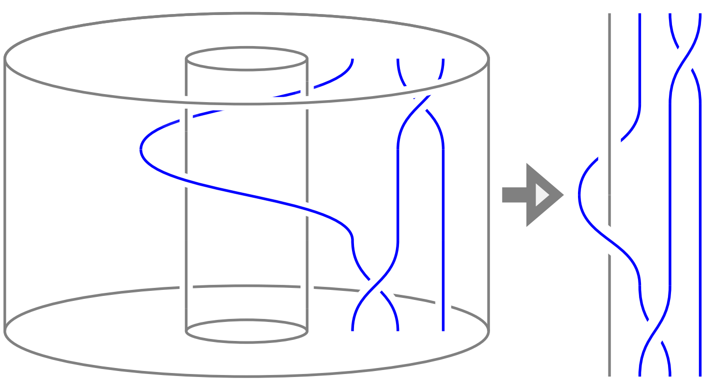
Abstract. Studying braids in an annulus rather than braids in a disk yields a theory of braidors, the analogue of Drinfeld associators. A priori, braidors appear weaker than associators and furthermore the equations make sense in more spaces than the associator equations do. However, computational evidence suggests the braidor equations are in fact equivalent to the associator equations. This suggests it may be useful to review the array of ideas related to associators, ie. Grothendieck-Teichmüller groups, multiple zeta values etc., in the simpler context of braidors in the hopes of gaining new information.

Associators. Recall that the theory of associators can be recast in the language of parenthesized braids. Let **PaB** be the (category/operad) created from parenthesized braids (ie. braids where the distance between strands matters) and **PaCD** be the category created from parenthesized chord diagrams. The data of a functor (operad morphism) Z : **PaB** → **PaCD** is essentially equivalent to the data of a Drinfeld associater in the Drinfeld-Kohno algebra $\mathfrak{t}_3$:



where $\Phi \in \mathfrak{t}_3$ is a Drinfeld associator.

Annular Braids.



The group of n-component braids in the annulus has presentation [B]

$$B_{1,n} = \left\langle \tau, \sigma_1, \cdots, \sigma_{n-1} \,\middle|\, \begin{matrix} \sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}, \\ [\sigma_i,\sigma_j]=1 \quad |i-j|>0, \\ [\tau,\sigma_1\tau\sigma_1]=1, \\ [\tau,\sigma_i]=1 \quad i>1, \end{matrix} \right\rangle$$

$B_{1,n}$ can equivalently [KP] be thought of as the subgroup of B_{n+1} whose first strand is pure, i.e. whose first strand ends in the first position again. In particular, $PB_{1,n} = PB_{n+1}$.

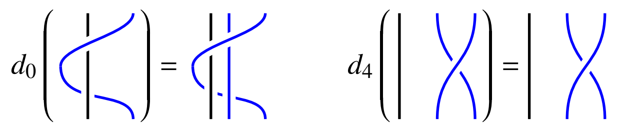
Definition 1. Let A be a commutative, associative $\mathbb{Q}$–algebra. The category $\boldsymbol{B_a}$ of braids in the annulus has objects the finite ordinals $\boldsymbol{n} = \{0, 1, \cdots, n\}$ and morphisms

$$\mathrm{Mor}(\boldsymbol{m}, \boldsymbol{n}) = \begin{cases} \varnothing & \boldsymbol{m} \neq \boldsymbol{n} \\ \left(P, \sum_{j=1}^k \beta_j B_j\right) & \boldsymbol{m} = \boldsymbol{n} \end{cases}$$

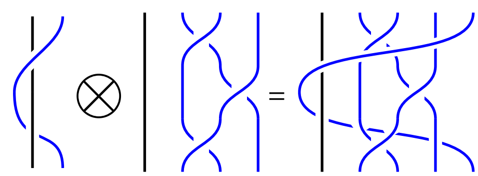
where P is a permutation in $S_{1,n}$ (ie. $P(0) = 0$,) each $B_j \in B_{1,n}$ is an n–component braid in the annulus with underlying permutation P and $\beta_j \in A$.

Operations. (These are functors $\boldsymbol{B_a} \to \boldsymbol{B_a}$)
Recall that the collection of all parenthesized braids $\{PaB_n\}_{n\in\mathbb{N}}$ has the structure of a cosimplicial set via the strand doubling and strand removal operations $\{d_i\}$ and $\{s_i\}$. In the annular case, the operations are

Extension: d_0 doubles the zeroth strand and d_{n+1} adds an identity strand to the right of the other strands.

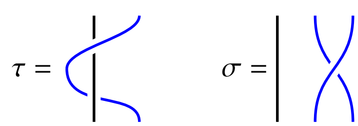


Monoidal Structure. Alternately, can view the collection of all parenthesized braids as the operad in groupoids whose ith partial composition is given by gluing a braid diagram into the ith strand of another one.
The analogous structure on $\boldsymbol{B_a}$ is that of a (strict) monoidal category with tensor product given by $\boldsymbol{m} \otimes \boldsymbol{n} = \boldsymbol{m} + \boldsymbol{n}$ on objects and on morphisms, $A_i \otimes B_j$ is obtained by gluing B_j into the core of A_i for any two annular braids A_i and B_j.



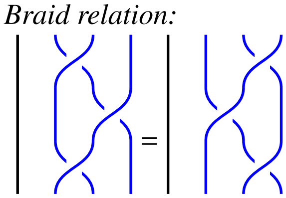
Coproduct. Define the coproduct □ by making any annular braid $B \in B_{1,n}$ grouplike.

Claim 1. *$\boldsymbol{B_a}$ is generated by repeated applications of d_0 and d_{n+1} (or alternatively as a strict monoidal category) by $\tau^{\pm 1}$ and $\sigma^{\pm 1}$ where*

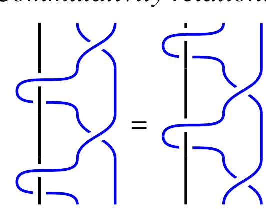


subject to the relations generated by

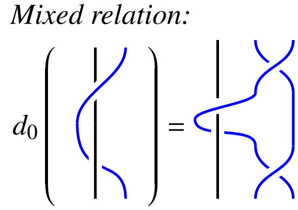
- *Braid relation:*



- *Commutativity relation:*



- *Mixed relation:*



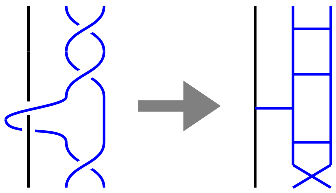
- *Locality in space and scale:*

Filtrations and Completions. $\boldsymbol{B_a}$ is the collection of all the group algebras $A[B_{1,n}]$ so many of the constructions possible in group algebras such as ideals, filtrations, quotients etc., can be extended to all of $\boldsymbol{B_a}$ by doing it for each $A[B_{1,n}]$ individually.

Let I be the augmentation ideal of $\boldsymbol{B_a}$, ie. the subcategory of all pairs $(P, \sum \beta_j B_j)$ such that $\sum \beta_j = 0$. The powers of I define the **unipotent filtration** $\mathcal{F}_m \boldsymbol{B_a} = I^{m+1}$ of $\boldsymbol{B_a}$.

Definition 2. Let $\boldsymbol{B_a}^{(m)} = \boldsymbol{B_a}/\mathcal{F}_m \boldsymbol{B_a}$ be the m**th unipotent quotient** of $\boldsymbol{B_a}$. Let $\widehat{\boldsymbol{B_a}} = \varprojlim_{m\to\infty} \boldsymbol{B_a}^{(m)}$ be the **unipotent completion** of $\boldsymbol{B_a}$.

Chord Diagrams for Annular Braids. A chord diagram for a braid remembers just the crossing information in a braid.



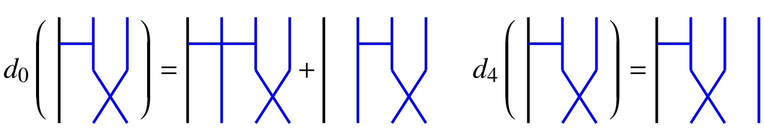
$= (1,2) \cdot t_{12} t_{02} t_{12} t_{12}$

Algebraically, the space of chord diagrams is the associated graded of the space of braids.

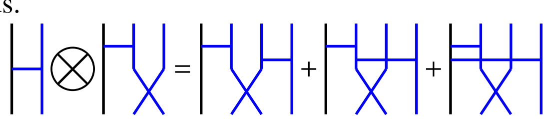
Definition 3. Let $\boldsymbol{CD_a} = \boldsymbol{CD_a}(A)$ be the category whose objects are finite ordinals and whose morphisms are A-linear combinations of formal products $D \cdot P$ where $P \in S_{1,n}$ is a permutation fixing the first element and $D \in \mathfrak{t}_{n+1}(A)$ is an element of the Drinfeld-Kohno algebra.

Structure. $\boldsymbol{CD_a}$ can be given all the same structure that was given to $\boldsymbol{B_a}$:

Doubling: d_0 doubles the zeroth strand in a chord diagram and sums over all ways of connecting chords. d_{n+1} adds an identity strand on the right.

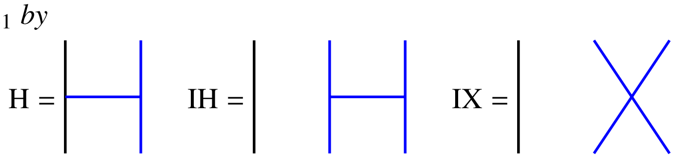


Monoidal Structure: $D \otimes C$ is obtained by gluing C into the zeroth strand of D and summing all ways of connecting chords.



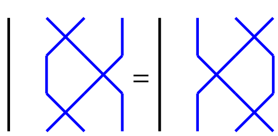
Coproduct: Define $\square$ by making any individual chord diagram primitive.

Claim 2. *$\boldsymbol{CD_a}$ is generated by repeated applications of d_0 and d_{n+1} by*

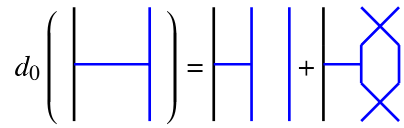


subject to the relations generated by

- *Braid relation:*



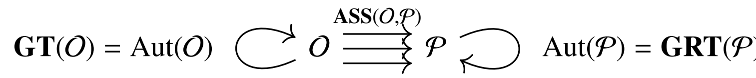
- *Semi-classical braid relation:*



- *Locality in space and scale.*

Filtrations and Completions. Define the unipotent filtration $\mathcal{F}_m \boldsymbol{CD_a} = I^{m+1}$, unipotent quotient $\boldsymbol{CD_a}^{(m)} = \boldsymbol{CD_a}/\mathcal{F}_m \boldsymbol{CD_a}$ and the unipotent completion $\widehat{\boldsymbol{CD_a}} = \varprojlim_{m\to\infty} \boldsymbol{CD_a}^{(m)}$.

General Setup.



Given isomorphic "algebraic structures" the groups $\mathbf{GT}(O) = \mathrm{Aut}(O)$ and $\mathbf{GRT}(\mathcal{P}) = \mathrm{Aut}(\mathcal{P})$ act simply and transitively on $\mathbf{ASS}(O, \mathcal{P})$, the set of isomorphisms $Z : O \to \mathcal{P}$, by pre and post-composition respectively.

Example. Let $O = \widehat{\mathbf{PaB}}$, $\mathcal{P} = \widehat{\mathbf{PaCD}}$. Then $\mathbf{ASS}(\widehat{\mathbf{PaB}}, \widehat{\mathbf{PaCD}})$ is the set of Drinfeld associators and $\mathbf{GT}(\widehat{\mathbf{PaB}}), \mathbf{GRT}(\widehat{\mathbf{PaCD}})$ are the (pro-unipotent versions of the) Grothendieck-Teichmüller groups as defined by Drinfeld [BN, D].

Braidors. A braidor will be essentially equivalent to a structure preserving functor $Z \in \mathbf{ASS}(\widehat{\boldsymbol{B_a}}, \widehat{\boldsymbol{CD_a}})$ where Z is required to preserve the underlying permutation of a braid and gr$Z$ must be the identity.

A structure preserving functor $Z : \widehat{\boldsymbol{B_a}} \to \widehat{\boldsymbol{CD_a}}$ is determined by the image of

$$Z(\sigma) = \mathrm{IX} \cdot B \qquad Z(\tau) = \mathrm{H} \cdot R$$

where $B \in \hat{\mathfrak{t}}_3$ and $R = \exp(t_{01})$. Ensuring the relations in $\boldsymbol{B_a}$ are satisfied yields:

Definition 4. A **braidor** is a grouplike (nondegenerate) element $B \in \hat{\mathfrak{t}}_3$ which satisfies the equations

$$B^{0,1,2} B^{02,1,3} B^{0,2,3} = B^{01,2,3} B^{0,1,3} B^{03,1,2} \qquad \text{(Braidor Eqn.)}$$
$$d_0(R) = R^{01,2} = B R^{0,2} B^{0,2,1} \qquad \text{(Mixed Eqn.)}$$
$$R^{0,1} B R^{0,2} B^{0,2,1} = B R^{0,2} B^{0,2,1} R^{0,1} \qquad \text{(Commutativity Eqn.)}$$

I will write **BRAID** for the collection of all braidors.

Grothendieck-Teichmüller Groups. An element $\phi \in \widehat{\mathbf{GRT}}_a$ is determined by

$$\phi(\mathrm{H}) = \mathrm{H} \cdot \Gamma_1, \quad \phi(\mathrm{IH}) = \mathrm{IH} \cdot \Gamma_2, \quad \phi(\mathrm{IX}) = \mathrm{IX} \cdot \Gamma_3$$

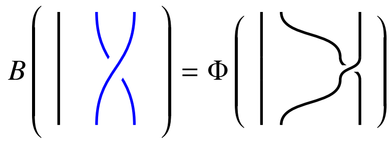
where $\Gamma_1 \in \hat{\mathfrak{t}}_2$ and $\Gamma_2, \Gamma_3 \in \hat{\mathfrak{t}}_3$. Further ϕ must respect the relations in Claim 2 and preserve all the structure of $\widehat{\mathbf{GRT}}_a$. This leads to the following definition

Definition 5. As a set, the group $\widehat{\mathbf{GRT}}_a$ is the collection of all grouplike nondegenerate triples $(\Gamma_1, \Gamma_2, \Gamma_3) \in \hat{\mathfrak{t}}_2 \times \hat{\mathfrak{t}}_3 \times \hat{\mathfrak{t}}_3$ which satisfy the equations

$$\Gamma_3^{0,1,2} \Gamma_3^{02,1,3} \Gamma_3^{0,3,2} = \Gamma_3^{01,2,3} \Gamma_3^{0,1,3} \Gamma_3^{03,1,2} \qquad \text{(Classical Braidor Eqn.)}$$
$$d_0(\Gamma_1) = \Gamma_2 + \Gamma_3 \Gamma_1^{0,2} \Gamma_3^{0,2,1} \qquad \text{(Semiclassical Braidor Eqn.)}$$

It is easy to see $\Gamma_1 = Id$ and the semiclassical braid equation determines Γ_2 in terms of Γ_1 and Γ_3 so it is really just a matter of finding Γ_3.

From Associators to Braidors. There is a map C : $\mathbf{ASS}(\widehat{\mathbf{PaB}}, \widehat{\mathbf{PaCD}}) \to \mathbf{BRAID}$ which sends any Drinfeld associator Φ to a braidor B via



ie. $B = \Phi e^{\frac{1}{2}t_{12}}\Phi^{-0,2,1}$. C is injective so $\mathbf{ASS}(\widehat{\mathbf{PaB}}, \widehat{\mathbf{PaCD}}) \hookrightarrow \mathbf{BRAID}$.

Results So Far.

Theorem 1. *Braidors can be constructed degree by degree (and rational braidors exist.)*

The proof of this theorem is inspired by and very similar to Drinfeld's proof of the same result for associators, especially as reformulated in [BN] although each step is simpler in the annular case.

Lemma 1. *The braid equation implies the semiclassical braid equation.*

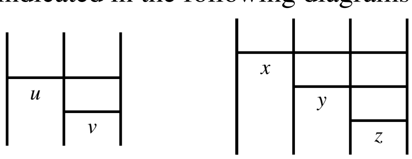
Proof. Assume $\Gamma_3 \in \mathfrak{t}_3$ satisfies the braidor and locality relations and let E be the error in the semiclassical braidor equation, ie.

$$1 + E = d_0(\Gamma_1) - \Gamma_2 - \Gamma_3\Gamma_1^{0,2}\Gamma_3^{0,2,1}.$$

The semiclassical braidor can be derived from the mixed relation in $\boldsymbol{B_a}$, so to find an equation satisfied by E, we need to find a relation between mixed relations as shown in Figure 1.
Each move in the diagram in which the mixed relation is applied will pick up an error E installed on the relevant strands. Comparing the errors along the two paths in the diagram, we get

$$E^{01,2,3} - E^{02,1,3} + E^{0,1,3} - E^{0,2,3} = 0. \tag{1}$$

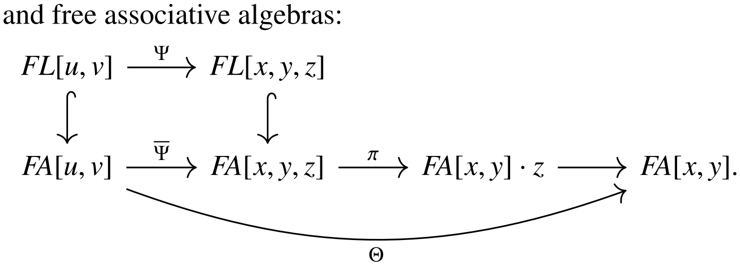
Consider the free Lie algebras $FL[u,v]$ and $FL[x,y,z]$ contained in $\mathfrak{t}_2$ and $\mathfrak{t}_3$ as indicated in the following diagrams:



Every term in Equation 1 lies in $FL[x,y,z]$ so upon restriction, E becomes a Lie polynomial $F \in \mathrm{Lie}[u,v]$ and Equation 1 becomes

$$F(x+y,z) - F(x+z,y) + F(x,y) - F(x,z) = 0 \tag{2}$$

Applying the Lie morphism $FL_3 \mapsto FL_2$, $\{x \mapsto x, y \mapsto y, z \mapsto 0\}$ to equation 2 implies $\langle F, u\rangle = 0$ while applying the Lie morphism $\{x \mapsto 0, y \mapsto y, z \mapsto z\}$ implies $F(u,v) - F(v,u) = \langle F, v\rangle(v-u)$, ie. that F is symmetric up to an error in degree 1.
We have the following diagram of maps between free lie algebras and free associative algebras:



Ψ, and $\overline{\Psi}$ are obtained by sending a series f to $f(x+y,z) - f(x+z,y) + f(x,y) - f(x,z)$ and π is the projection onto words ending in z. Notice that $\ker\Psi \subseteq \ker\Theta$.
Regarding $F \in FL[u,v] \subseteq FA[u,v]$ as an element in the algebra of noncommutative series in u and v, write $F(u,v) = F_1(u,v)u + F_2(u,v)v$. Symmetry of F implies $F_1(u,v) = F_2(v,u) - \langle F, v\rangle$.
Tracing through the maps, $\Theta F = 0$ is equivalent to

$$F_2(x,y) = \sum_{n=0}^{\infty}\langle F_2, u^n\rangle\Big((x+y)^n - y^n\Big) - \langle F, v\rangle$$

and hence

$$F(u,v) = \sum_{n=0}^{\infty}\langle F_2, u^n\rangle\Big(\big[(u+v)^n - u^n\big]u + \big[(u+v)^n - v^n\big]v\Big) - \langle F, v\rangle v \tag{3}$$

In order for $F(u,v)$ to be a Lie polynomial, F must be primitive but it is clear from equation 3 that primitivity implies $F_2 = 0$. Thus $F(u,v) = \langle F, v\rangle v$ and since braidors are fixed in degree 1, $F = 0$. □

Lemma 2. *The natural homomorphism* $\widehat{\mathbf{GRT}}_a^{(m)} \to \widehat{\mathbf{GRT}}_a^{(m-1)}$ *is surjective.*

Proof. Since $\widehat{\mathbf{GRT}}_a^{(m)}$ are connected reduced algebraic group schemes, it is enough to prove the statement on the level of Lie algebras, defined by the linearization

$$\gamma^{0,1,2} + \gamma^{02,1,3} + \gamma^{0,3,2} = \gamma^{01,2,3} + \gamma^{0,1,3} + \gamma^{03,1,2}.$$

But clearly if γ is of degree m and satisfies this, it can be extended by taking the $m+1$st degree component to be 0 for example. □

Lemma 3. *The natural map* $\mathbf{BRAID}^{(m)} \to \mathbf{BRAID}^{(m-1)}$ *is surjective.*

Proof. Since $\mathbf{ASS}(\widehat{\mathbf{PaB}}, \widehat{\mathbf{PaCD}}) \hookrightarrow \mathbf{BRAID}$, there exists at least one braidor, so there is one degree $m-1$ braidor which extends to degree m. Since $\widehat{\mathbf{GRT}}_a^{(m-1)}$ acts transitively all degree $m-1$ braidors must extend by the previous lemma. □

Other Spaces. Some useful techniques (the so-called "cubical complex of a permutation group representation" [SW]) apply to spaces related to the Drinfeld-Kohno algebra but not $\mathfrak{t}_n$ itself. For example, $\mathfrak{t}_3$ embeds in the Kashiwara-Vergne lie algebra $\mathfrak{krv}_3$ and these techniques work here so it may be useful to study the theory of braidors in this and other relevant spaces.

Conjectures/Computational Evidence.

Conjecture 1. *The braid equation implies the commutativity and mixed equations.*

- *Verified to degree 8 in the Drinfeld-Kohno algebra.*

Conjecture 2. $C : \mathbf{ASS}(\widehat{\mathbf{PaB}}, \widehat{\mathbf{PaCD}}) \to \mathbf{BRAID}$ *is a bijection.*

- *Up to degree 10, the dimension of the space of braidors is equal to that of associators.*

Conjecture 3. $\widehat{\mathbf{GT}}_a \cong \widehat{\mathbf{GT}}$ *and* $\widehat{\mathbf{GRT}}_a \cong \widehat{\mathbf{GRT}}$.

Conjecture 4. *Braidors in* TAut *and* SAut *generally fail to extend but braidors in* KV *do extend. Furthermore all braidors in* KV *come from braidors in the Drinfeld-Kohno algebra.*

- *Verified up to degree* 8.

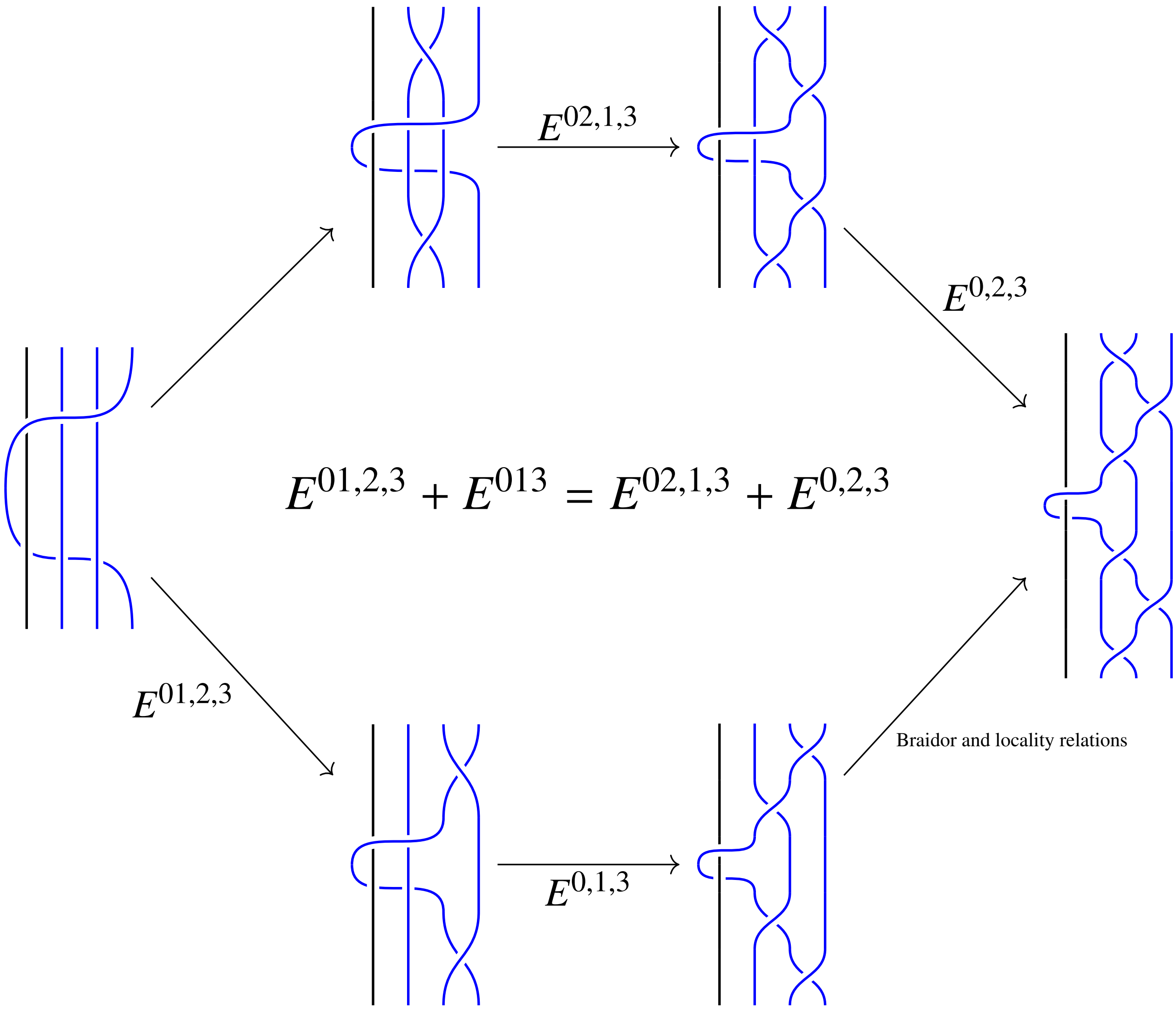


Figure 1. The equation satisfied by E is obtained by going from left to right along the two paths and keeping track of the error produced in the semiclassical braidor by each application of the mixed relation.

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