A 3-Dimensional Perspective on Drinfel’d’s Theory of Quasi-Hopf Algebras
California Institute of Technology, February 21, 2000
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Quasi-Hopf algebra (Drinfel’d): \((A, m, \Delta, \Phi)\) s.t. \((\text{typ. } A = \hat{U}(g))\)

\[ m : A \otimes A \rightarrow A, \quad \Delta : A \rightarrow A \otimes A, \quad \Phi \in A \otimes A \otimes A, \]
\[
(I \otimes \Delta)(\Delta(a)) = \Phi \cdot (\Delta \otimes I)(\Delta(a)) \cdot \Phi^{-1}, \quad a \in A,
\]
\[
(I \otimes I \otimes \Delta)(\Phi) \cdot ((\Delta \otimes I)(\Phi)) = (I \otimes \Phi) \cdot (I \otimes \Delta \otimes I)(\Phi) \cdot (\Phi \otimes I)
\]

(\text{“the pentagon \(\bigcirc\)”})

\(+\text{more axioms…}\)

Why? It makes \(\text{Rep}(A)\) a tensor category; \(\Delta\) defines \(M_1 \otimes M_2\), \(\Phi\) defines a map \((M_1 \otimes M_2) \otimes M_3 \rightarrow M_1 \otimes (M_2 \otimes M_3)\), and \(\odot\) ensures:

\[
(M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \quad (M_1 \otimes (M_2 \otimes M_3)) \otimes M_4
\]

\(M_1 \otimes (M_2 \otimes (M_3 \otimes M_4)) \quad M_1 \otimes ((M_2 \otimes M_3) \otimes M_4)
\]

Politically incorrect: this view is harmful to the categorically challenged!

What is it good for? E.g., constructing link invariants

\[
Z(\gamma) = \int_{\text{\gamma-connections}} \mathcal{D}A \, \text{hol}_c(A) \exp \left[ \frac{i k}{4 \pi} \int_{\mathbb{R}^3} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right]
\]

ribbon link \(\gamma \mapsto \text{hol}_c(A)\) is \(\{\text{in } G \text{ dual to reps in } \hat{U}(g)\}\)

THE ONE THING TO REMEMBER:
Embedded Trivalent ribbon Graph \(\gamma\)
Easy, powerful moves:

Using moves, ETG is generated by ribbon twists and the tetrahedron

\[
\begin{array}{c}
\text{blue: blueprint} \\
\text{red: computed}
\end{array}
\]

Modulo the relation(s):

\[
\begin{array}{c}
\text{Claim: } \hat{U}(\Delta) \equiv \hat{U}(g)^{\otimes 3}/g \quad \text{and under this isomorphism the above relation for } Z(\Delta) \text{ becomes the pentagon } \bigcirc \text{ for } \Phi, \quad \text{and likewise for all other Drinfel’d’s axioms.}
\end{array}
\]

Proof: Collapse a tree:

Why am I happy?
1. 3-dimensional picture of associators (\(\Phi\)’s).
2. A direct link between CSW and quasi-Hopf.
3. Will have applications...

Joint with Dylan Thurston

This handout is at
http://www.ma.huji.ac.il/~drorbn/Talks/CalTech-000221