On double shuffle relations for MZVs

collaboration with H. Furusho (Nagoya)

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Papers

- **The Betti side of the double shuffle theory. III. Double shuffle relations for associators**, in preparation.
Section 0: The context
Two approaches to the algebraic relations between MZVs

- **based on combinatorics**: The MZVs satisfy double shuffle relations (Ihara-Kaneko-Zagier 2006, Racinet 2002).

- **based on the geometry of moduli space of curves**: The "KZ associator" (Drinfeld) is a generating series for MZVs (Le-Murakami 1996). It satisfies algebraic relations (Drinfeld 1991).
The context

**Relations between the two approaches**

**Thm** (Furusho 2011, Deligne-Terasoma 2005 (announcement)). The associator relations imply the double shuffle relations.

- Ideas of (Furusho 2011): Associator relations take place in $U_{p_5}$. Construction of explicit linear forms on $U_{p_5}$, based on multiple polylogs. Combinatorics of linear forms.

- Ideas of (Deligne-Terasoma 2005): Geometric constructions with moduli spaces $M_{0,4}$ and $M_{0,5}$. Perverse sheaves on these spaces. Redaction is still unfinished.

**Remark:** [Hirose-Sato 2018+] and [Furusho 2018+] give another proof.

**Today:** New proof of theorem based on Deligne-Terasoma ideas.
Detailed plan: (1)

(1) The double shuffle formalism

(1a) MZVs

(1b) Examples of double shuffle relations

(1c) The double shuffle formalism:
   - algebra $\mathcal{W}^{\text{DR}}$ and coproduct $\Delta^\mathcal{W}$ on it (harmonic coproduct)
   - a rank 1 module $\mathcal{M}^{\text{DR}}$ over it and a coproduct $\Delta^\mathcal{M}$ over this module
   - $\Gamma$-functions $\Gamma_\Phi(t)$

(1d) Formulation of double shuffle relation in terms of double shuffle formalism.
(2) Two comparison results and the main result

(2a) Betti version \((\mathcal{W}^B, \Delta^W, \mathcal{M}^B, \Delta^M)\) of \((\mathcal{W}^{DR}, \Delta^W, \mathcal{M}^{DR}, \Delta^M)\)

(2b) "comparison" operators \(\text{comp}^{(1),\mathcal{W}}_{(\mu,\Phi)} : \mathcal{W}^{DR} \to \mathcal{W}^B\) and \(\text{comp}^{(10),\mathcal{M}}_{(\mu,\Phi)} : \mathcal{M}^{DR} \to \mathcal{M}^B\).

(2c) comparison results:
(algebra): for \((\mu, \Phi)\) associator, \(\text{comp}^{(1),\mathcal{W}}_{(\mu,\Phi)}\) brings \(\Delta^W\) to \(\Delta^\#\)
(module): for \((\mu, \Phi)\) associator, \(\text{comp}^{(10),\mathcal{M}}_{(\mu,\Phi)}\) brings \(\Delta^\#\) to \(\Delta^\#\)

(2d) why the "module" comparison result implies the associator-double shuffle implication (main result).
(3) Proof of "algebra" comparison result

(3a) Interpretation of $\Delta^W$ in terms of moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$ (Deligne-Terasoma)

(3b) Interpretation of $\Delta^\#_W$ in terms of moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$

(3c) Proof of comparison result based on study of $\text{comp}_{(\mu, \Phi)} : \text{PaB} \to \text{PaCD}$ evaluated at $(((\ddots))\cdot)$
(4) Detailed plan: (4)

(4) Proof of "module" comparison result

(4a) Interpretation of $\Delta^M_\star$ in terms of moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$

(4b) Interpretation of $\Delta^M_\#$ in terms of moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$

(4c) Proof of comparison result based on study of $\text{comp}_{(\mu,\Phi)} : PaB \rightarrow \widehat{PaCD}$ evaluated at $(\bullet(\bullet\bullet))\bullet$
Section 1: The double shuffle formalism
The double shuffle formalism

Multiple Zeta Value (MZV)

For $k_1, \ldots, k_{m-1} \geq 1$ and $k_m \geq 2$,

$$\zeta(k_1, \ldots, k_m) := \sum_{0 < n_1 < \cdots < n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \in \mathbb{R} : \text{MZV}$$

- The sum converges iff $k_m > 1$.
- $m = 1$: Riemann zeta value $\zeta(k)$.
- $m = 2$: Double zeta value by Goldbach and Euler.

Double Shuffle relations for MZV’s

‘=’ Shuffle + Harmonic product
Shuffle product:

e.g. \[ \zeta(a) \zeta(b) = \int_0^{s_1 < \ldots < s_a < 1} \frac{ds_1}{1-s_1} \wedge \frac{ds_2}{s_2} \wedge \cdots \wedge \frac{ds_a}{s_a} \times \int_0^{t_1 < \ldots < t_b < 1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_b}{t_b} \]

\[ = \sum \int_0^1 \text{all shuffles} \]

\[ = \sum_{i+j=a+b} \left\{ (i-1) + (j-1) \right\} \zeta(i, j) \]

Harmonic product:

e.g. \[ \zeta(a) \zeta(b) = \sum_{0<k} \frac{1}{k^a} \cdot \sum_{0<l} \frac{1}{l^b} = \left( \sum_{0<k<l} + \sum_{0<k=l} + \sum_{0<l<k} \right) \frac{1}{k^a l^b} \]

\[ = \zeta(a, b) + \zeta(a + b) + \zeta(b, a). \]
The double shuffle formalism (Racinet)

\( \mathcal{V}^{\text{DR}} \) := \( \mathbb{C}\langle e_0, e_1 \rangle \) : free graded algebra over \( e_0, e_1 \) of deg=1.

Coproduct \( \Delta : \mathcal{V}^{\text{DR}} \to (\mathcal{V}^{\text{DR}})^{\otimes 2}, \ e_i \mapsto e_i \otimes 1 + 1 \otimes e_i. \)

Subalgebra \( \mathcal{W}^{\text{DR}} := \mathbb{C} \oplus \mathcal{V}^{\text{DR}} e_1 (\hookrightarrow \mathcal{V}^{\text{DR}}). \)

Presentation: \( \mathcal{W}^{\text{DR}} \) is freely generated by \( y_1, y_2, \ldots \), where \( y_n := -e_0^{n-1} e_1. \)

Harmonic coproduct \( \Delta^{\mathcal{W}}_\star : \mathcal{W}^{\text{DR}} \to (\mathcal{W}^{\text{DR}})^{\otimes 2}, \)

\[ \Delta^{\mathcal{W}}_\star(y_n) = y_n \otimes 1 + 1 \otimes y_n + \sum_{k+l=n} y_k \otimes y_l \]

equips \( \mathcal{W}^{\text{DR}} \) with Hopf algebra structure.
Quotient $\mathcal{M}^{\text{DR}} := V^{\text{DR}} / V^{\text{DR}} e_0$ and the canonical projection $\text{can} : V^{\text{DR}} \rightarrow \mathcal{M}^{\text{DR}}$. Then $\mathcal{M}^{\text{DR}}$ is a free $\mathcal{V}^{\text{DR}}$-module of rank 1, generated by $1_{\text{DR}} := \text{projection of } 1 \in V^{\text{DR}}$. Define $\Delta_{\mathcal{M}}^{\ast} : \mathcal{M}^{\text{DR}} \rightarrow (\mathcal{M}^{\text{DR}})^{\otimes 2}$ as the transport of $\Delta_{\mathcal{W}}^{\ast}$ under the isomorphism $\mathcal{W}^{\text{DR}} \rightarrow \mathcal{M}^{\text{DR}}$ induced by action on $1_{\text{DR}}$.

Notation:

For $\Phi \in \hat{\mathcal{V}}^{\text{DR}} := \mathbb{C}\langle\langle e_0, e_1 \rangle\rangle$, set

$$
\Gamma_{\Phi}(-e_1)^{-1} := \exp\left(\sum_{n \geq 1} \frac{1}{n} (\Phi|e_0^{n-1} e_1)e_1^n\right) \in \hat{\mathcal{V}}^{\text{DR}},
$$

$$
\Phi_{\ast} := \text{can}(\Gamma_{\Phi}(-e_1)^{-1} \Phi) \in \hat{\mathcal{M}}^{\text{DR}}.
$$
Generating series of MZVs:

\[ \Phi_{KZ} := 1 + \sum (-1)^m \zeta(k_1, \ldots, k_m) \cdot e_0^{k_m-1} e_1 \cdots e_0^{k_1-1} e_1 \]

\[ + \text{terms in } e_1 \mathcal{V}^{DR} + \mathcal{V}^{DR} e_0 \in \hat{\mathcal{V}}^{DR}. \]

Relations:

shuffle relation: \( \hat{\Delta}(\Phi_{KZ}) = \Phi_{KZ} \otimes \Phi_{KZ} \) (relation in \( \hat{\mathcal{V}}^{DR})^{\otimes 2} \)

harmonic relation: \( \hat{\Delta}^M(\Phi_{KZ, \star}) = \Phi_{KZ, \star} \otimes \Phi_{KZ, \star} \) (relation in \( \hat{\mathcal{M}}^{DR})^{\otimes 2} \)).

One says that the collection of commutative variables \( \zeta^f(k_1, \ldots, k_m) \) satisfy the double shuffle relations iff they satisfy the above relations with \( \Phi_{KZ} \) replaced by

\[ \Phi := 1 + \sum (-1)^m \zeta^f(k_1, \ldots, k_m)(e_0^{k_m-1} e_1 \cdots e_0^{k_1-1} e_1 \]

\[ + \text{terms in } e_1 \mathcal{V}^{DR} + \mathcal{V}^{DR} e_0. \]
Section 2: The comparison results and the main result
Betti version \( (\mathcal{W}^B, \Delta_\#^W, \mathcal{M}^B, \Delta_\#^M) \) of \( (\mathcal{W}^{DR}, \Delta_*^W, \mathcal{M}^{DR}, \Delta_*^M) \)

- Algebra \( \mathcal{V}^B := \mathbb{C}F_2 \), where \( F_2 := \) free group over \( X_0, X_1 \).

  Coproduct \( \Delta : X_0, X_1 \) are group-like.

- Subalgebra \( \mathcal{W}^B := \mathbb{C} \oplus \mathcal{V}^B(X_1 - 1) (\hookrightarrow \mathcal{V}^B) \).

  Presentation: generators \( X_1, X_{-1}^1, Y_{n}^\pm := (X_{0}^\pm - 1)^{n-1}X_{0}^\pm (1 - X_{1}^\pm) (n \geq 1) \)

  with only relations \( X_1 \cdot X_{-1}^1 = X_{-1}^1 \cdot X_1 = 1 \).

  Coproduct on \( \mathcal{W}^B \) is \( \Delta_\#^W : \mathcal{W}^B \to (\mathcal{W}^B)^{\otimes 2} \) given by

  \[
  \Delta_\#^W(X_{1}^{\pm 1}) = X_{1}^{\pm 1} \otimes X_{1}^{\pm 1},
  \]

  \[
  \Delta_\#^W(Y_{n}^\pm) = Y_{n}^\pm \otimes 1 + 1 \otimes Y_{n}^\pm + \sum_{k+l=n} Y_{k}^\pm \otimes Y_{l}^\pm
  \]

  equips \( \mathcal{W}^B \) with a Hopf algebra structure.
Quotient vector space $\mathcal{M}^B := \mathcal{V}^B / \mathcal{V}^B(X_0 - 1)$. Set $1_B := $ projection of $1 \in \mathcal{V}^B$. Then $\mathcal{M}^B$ is a free $\mathcal{W}^B$-module generated by $1_B$. Define

$$\Delta^\mathcal{M}_\# : \mathcal{M}^B \to (\mathcal{M}^B)^{\otimes 2}$$

as the transport of $\Delta^\mathcal{W}_\# : \mathcal{W}^B \to (\mathcal{W}^B)^{\otimes 2}$ under the isomorphism $\mathcal{W}^B \to \mathcal{M}^B$ induced by action on $1_B$. 
Comparison operator $\text{comp}_{(\mu,\Phi)}^{(1),\mathcal{W}} : \mathcal{W}^{\text{DR}} \to \mathcal{W}^{\text{B}}$

Algebra isomorphisms: For $(\mu, \Phi) \in \mathbb{C}^\times \times \mathcal{G}(\hat{\mathcal{V}}^{\text{DR}})$ (notation: $\mathcal{G}(\mathcal{G})=\text{group of group-like elements of a Hopf algebra}$), define algebra isomorphism

$$\text{comp}_{(\mu,\Phi)}^{(1),\mathcal{V}} : \hat{\mathcal{V}}^{\text{B}} \to \hat{\mathcal{V}}^{\text{DR}},$$

$$X_0 \mapsto \Phi \cdot \exp(\mu e_0) \cdot \Phi^{-1}, \quad X_1 \mapsto \exp(\mu e_1).$$

Note: when $(\mu, \Phi) = (2\pi i, \Phi_{\text{KZ}})$, this is the period isomorphism $\mathbb{C}\pi_1^\text{B}(\mathcal{M}_{0,4}, \vec{1})^\wedge \to \mathbb{C}\pi_1^{\text{DR}}(\mathcal{M}_{0,4}, \vec{1})^\wedge$ between the Betti and De Rham fundamental group algebras.

The isomorphism $\text{comp}_{(\mu,\Phi)}^{(1),\mathcal{V}}$ restricts to an algebra isomorphism

$$\text{comp}_{(\mu,\Phi)}^{(1),\mathcal{W}} : \hat{\mathcal{W}}^{\text{B}} \to \hat{\mathcal{W}}^{\text{DR}}.$$
Comparison operator \( \text{comp}_{\mu, \Phi}^{(10), \mathcal{M}} : \mathcal{M}^{\text{DR}} \to \mathcal{M}^{\text{B}} \)

Module isomorphisms: For \((\mu, g)\) in \(\mathbb{C}^\times \times \mathcal{G}^{\hat{\mathcal{V}}_{\text{DR}}}\), define

\[
\text{comp}_{\mu, \Phi}^{(10), \mathcal{V}} : \hat{\mathcal{V}}^{\text{B}} \to \hat{\mathcal{V}}_{\text{DR}}, \quad \nu \mapsto \text{comp}_{\mu, \Phi}^{(1), \mathcal{V}}(\nu) \cdot \Phi.
\]

Note: when \((\mu, \Phi) = (2\pi i, \Phi_{\text{KZ}})\), this is the period isomorphism \(\mathbb{C} \pi^B_1(\mathcal{M}_{0,4}, \vec{1}, \vec{0})^\wedge \to \mathbb{C} \pi^\text{DR}_1(\mathcal{M}_{0,4}, \vec{1}, \vec{0})^\wedge\) between the Betti and De Rham fundamental groupoid modules.

This isomorphism factors to an isomorphism

\[
\text{comp}_{\mu, \Phi}^\mathcal{M} : \hat{\mathcal{M}}^{\text{B}} \to \hat{\mathcal{M}}_{\text{DR}}.
\]
Summary

We summarize the situation as follows:

<table>
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<th>algebras</th>
<th>modules over $\mathcal{V}^{B/DR}$</th>
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<tr>
<td>morphisms</td>
<td>$\mathcal{W}^{B/DR}$ ↔ $\mathcal{V}^{B/DR}$</td>
<td>$\mathcal{V}^{B/DR}$ ↪ $\mathcal{M}^{B/DR}$</td>
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<td>coproduct</td>
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<td>$\Delta / \Delta$</td>
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<td>fake B/DR isoms</td>
<td>$\text{comp}^{\mathcal{W}}_{(\mu, \Phi)}$</td>
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</tr>
<tr>
<td>geometry</td>
<td>$\pi_1(\mathcal{M}_{0,4} ; \vec{1})$</td>
<td>$\pi_1(\mathcal{M}_{0,4} ; \vec{1}, \vec{0})$</td>
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Definition: (Drinfeld 1991, Furusho 2010)

An **associator** is a pair \((\mu, \Phi) \in \hat{G}_{DR}^\times = \mathbb{C}^\times \times (\hat{\mathcal{V}}_{DR})^\times\)

(recall that \((\hat{\mathcal{V}}_{DR})^\times = \mathbb{C}(\langle e_0, e_1 \rangle)^\times\)) such that

- \((\Phi|e_0) = (\Phi|e_1) = 0,\)
- \((\Phi|e_0e_1) = \mu^2/24,\)
- \(\hat{\Delta}(\Phi) = \Phi \otimes \Phi,\)
- \(\Phi^{345} \Phi^{512} \Phi^{234} \Phi^{451} \Phi^{123} = 1 \text{ in } (U\mathfrak{g}_5)^\wedge.\)

**Example:** \((\mu, \Phi) = (2\pi i, \Phi_{KZ})\) is an associator.
Main property of associators (Drinfeld 1991, Bar-Natan 1998)

An associator \((\mu, \Phi)\) gives rise to a functor

\[
\text{comp}_{(\mu, \Phi)} : \text{PaB} \to \text{PaCD}
\]

between the categories of parenthesized braids and parenthesised chord diagrams.

Specializing this functor to sets of morphisms, one gets a system of isomorphisms of topological vector spaces

\[
\text{comp}^{(n), \vec{a}, \vec{b}}_{(\mu, \Phi)} : \mathbb{C}\pi^B_1(\mathcal{M}_{0,n}; \vec{a}, \vec{b}) \to \mathbb{C}\pi^{\text{DR}}_1(\mathcal{M}_{0,n}; \vec{a}, \vec{b})
\]

where \(\vec{a}, \vec{b}\) are tangential base points of \(\mathcal{M}_{0,n}\).

**Particular cases:**

\[
\text{comp}^{(1)}_{(\mu, \Phi)} (n = 4, (\vec{a}, \vec{b}) = (\vec{1}, \vec{1})),
\]

\[
\text{comp}^{(10)}_{(\mu, \Phi)} (n = 4, (\vec{a}, \vec{b}) = (\vec{1}, \vec{0})).
\]
"Algebra" comparison result

If \((\mu, \Phi)\) is an associator, then the following diagram commutes:

\[
\begin{align*}
\hat{\mathcal{W}}^B & \xrightarrow{\Delta^W_{\#}} (\hat{\mathcal{W}}^B)^{\otimes 2} \\
\hat{\mathcal{W}}^{DR} & \xrightarrow{\hat{\Delta}^W} (\hat{\mathcal{W}}^{DR})^{\otimes 2} & (\hat{\mathcal{W}}^{DR})^{\otimes 2} & \xrightarrow{\text{Ad}(B_{\Phi})} (\hat{\mathcal{W}}^{DR})^{\otimes 2}
\end{align*}
\]

Here

\[
B_{\Phi} := \frac{\Gamma_{\Phi}(-e_1 \otimes 1)\Gamma_{\Phi}(-1 \otimes e_1)}{\Gamma_{\Phi}(-e_1 \otimes 1 - 1 \otimes e_1)} \in ((\hat{\mathcal{W}}^{DR})^{\otimes 2})^\times.
\]

The proof of this result will be an ingredient in the proof of the next result:
"Module" comparison result

If \((\mu, \Phi)\) is an associator, then the following diagram commutes

\[
\begin{array}{ccc}
\hat{M}^B & \xrightarrow{\hat{\Delta}^M_\#} & (\hat{M}^B)^{\otimes 2} \\
\text{comp}^M_{(\mu, \Phi)} \downarrow & \simeq & \downarrow (\text{comp}^M_{(\mu, \Phi)})^{\otimes 2} \\
\hat{M}^{DR} & \xrightarrow{\hat{\Delta}^M} & (\hat{M}^{DR})^{\otimes 2} \\
\hat{\Delta}_\star & \xrightarrow{B_\Phi \cdot (-)} & (\hat{M}^{DR})^{\otimes 2}
\end{array}
\]

Why the "module" comparison result implies the associator-double shuffle implication (main result)?

Apply to \(1_B \in \hat{M}^B: \hat{\Delta}^M_\#(1_B) = 1_B^{\otimes 2}\).

Then \(\text{comp}^M_{(\mu, \Phi)}(1_B) = \Phi \cdot 1_{DR} = \text{can}(\Phi)\).

So \(\Delta^M_\star(\text{can}(\Phi)) = B_\Phi^{-1} \cdot \text{can}(\Phi)^{\otimes 2}\). Hence \(\Delta^M_\star(\Phi_\star) = \Phi^{\otimes 2}\). □
Section 3: Proof of "algebra" comparison result
Interpretation of $\Delta^W$ in terms of moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$

(Deligne-Terasoma)

Recall that

$$\mathcal{V}_{DR} \simeq ULie\pi_{1}^{DR}(\mathcal{M}_{0,4}; t_{1}).$$

Set:

- $\mathcal{V}^{DR}(\mathcal{M}_{0,5}) := U\mathfrak{H}_{5} \simeq ULie\pi_{1}^{DR}(\mathcal{M}_{0,5}; t_{11}) \overset{\ell}{\leftarrow} \mathcal{V}^{DR}$
  - $e_{23}, e_{12} \leftarrow e_{0}, e_{1}$
  - $\ell$ = algebra morphism

- $pr_{i} : \mathcal{M}_{0,5} \twoheadrightarrow \mathcal{M}_{0,4}$ \hspace{1cm} ($i = 1, \ldots, 5$)
  - $pr_{i} : \mathcal{V}^{DR}(\mathcal{M}_{0,5}) \twoheadrightarrow \mathcal{V}_{DR}$
  - $pr_{i}$ are algebra morphisms and $pr_{5} \circ \ell = id$

- $pr_{12} : \mathcal{V}^{DR}(\mathcal{M}_{0,5}) \rightarrow (\mathcal{V}_{DR})^\otimes 2$ defined by
  - $pr_{12} := (pr_{1} \otimes pr_{2}) \circ \Delta$. 

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\[ \mathcal{V}^{\text{DR}}(\mathcal{M}_{0,5}) \cong \ker\{\mathcal{V}^{\text{DR}}(\mathcal{M}_{0,5}) \xrightarrow{\text{pr}_5} \mathcal{V}^{\text{DR}}\} \cong \mathcal{V}^{\text{DR}}(\mathcal{M}_{0,5})^\oplus 3 \]

\[ \Rightarrow \varpi : \mathcal{V}^{\text{DR}}(\mathcal{M}_{0,5}) \rightarrow M_3(\mathcal{V}^{\text{DR}}(\mathcal{M}_{0,5})) \]

\(\varpi\) is an algebra morphism.

**row**\(_1\) := \((e_1 \otimes 1, -1 \otimes e_1, 0) \in M_{1 \times 3}( (\mathcal{V}^{\text{DR}})^\otimes 2) \)

**col**\(_1\) := \[
\begin{pmatrix}
1 \otimes 1 \\
-1 \otimes 1 \\
0
\end{pmatrix} \in M_{3 \times 1}( (\mathcal{V}^{\text{DR}})^\otimes 2)
\]

Define an algebra morphism

\[ \rho : \mathcal{V}^{\text{DR}} \rightarrow M_3((\mathcal{V}^{\text{DR}})^\otimes 2) \]

as \[ \mathcal{V}^{\text{DR}} \xrightarrow{\ell} \mathcal{V}^{\text{DR}}(\mathcal{M}_{0,5}) \xrightarrow{\varpi} M_3(\mathcal{V}^{\text{DR}}(\mathcal{M}_{0,5})) \xrightarrow{M_3(\text{pr}_{12})} M_3((\mathcal{V}^{\text{DR}})^\otimes 2). \]
Proposition: The following diagram commutes

\[
\begin{array}{ccc}
\mathcal{V}_{\text{DR}} & \xrightarrow{\rho} & M_3((\mathcal{V}_{\text{DR}})^{\otimes 2}) & \xrightarrow{\text{row}_1 \cdot (-) \cdot \text{col}_1} & (\mathcal{V}_{\text{DR}})^{\otimes 2} \\
\downarrow \scriptstyle{(-) \cdot e_1} & & \downarrow \scriptstyle{\text{Ad}((e_1^{-1})^{\otimes 2})} & & \\
\mathcal{W}_{\text{DR},+} & & (\mathcal{V}_{\text{DR}})^{\otimes 2} & & (\mathcal{W}_{\text{DR}})^{\otimes 2} \\
\downarrow \scriptstyle{\Delta_{\mathcal{W}}^W} & & & & \\
\end{array}
\]
Proof. (a) Define algebra \((\mathcal{V}^{\text{DR}}, \cdot_{e_1})\) by
\[
a \cdot_{e_1} b := a \cdot e_1 \cdot b.
\]
(b) Show that \(\rho(e_1) = \col_1 \cdot \row_1\) and derive that
\[
(\mathcal{V}^{\text{DR}}, \cdot_{e_1}) \xrightarrow{\rho} M_3((\mathcal{V}^{\text{DR}} \otimes 2)) \xrightarrow{\row_1 \cdot (-) \cdot \col_1} (\mathcal{V}^{\text{DR}} \otimes 2)\]
is an algebra morphism.
(c) The map \((\mathcal{V}^{\text{DR}}, \cdot_{e_1}) \xrightarrow{(-) \cdot e_1} \mathcal{W}_+^{\text{DR}}\) is also an algebra morphism.
(d) So if \(\mathcal{V}^{\text{DR}}\) is equipped with \(\cdot_{e_1}\), all maps in diagram are algebra morphisms.
(d) Prove commutativity on each \(e_0^n\) by direct computation.
(e) Conclude from fact that \((\mathcal{V}^{\text{DR}}, \cdot_{e_1})\) is algebra-generated by the \(e_0^n, n \geq 0\).
Recall that $V^B = \mathbb{C}\langle X_0^{\pm1}, X_1^{\pm1} \rangle = \mathbb{C}F_2 \simeq \mathbb{C}\pi_1^{\text{topo}}(\mathcal{M}_{0,4}; \vec{1})$,

$$W^B = \mathbb{C} \oplus V^B \cdot (X_1 - 1).$$

Set:

- $V^B(\mathcal{M}_{0,5}) := \mathbb{C}P_5^* \simeq \mathbb{C}\pi_1^{\text{topo}}(\mathcal{M}_{0,5}; t_{11}) \overset{\ell}{\hookleftarrow} V^B$
  
  \[ x_{23}, x_{12} \leftrightarrow X_0, X_1 \]

$\ell$ is an algebra morphism.

- $\text{pr}_i : \mathcal{M}_{0,5} \twoheadrightarrow \mathcal{M}_{0,4}$ \hspace{1em} ($i = 1, \ldots, 5$)
  
  $\Rightarrow \text{pr}_i : V^B(\mathcal{M}_{0,5}) \twoheadrightarrow V_B$

$\text{pr}_i$ is an algebra morphism and $\text{pr}_5 \circ \ell = \text{id}$

$\text{pr}_{12} := (\text{pr}_1 \otimes \text{pr}_2) \circ \Lambda.$
\( \mathcal{V}^B(M_{0,5}) \cong \ker \{ \mathcal{V}^B(M_{0,5}) \xrightarrow{\text{pr}_5} \mathcal{V}^B \} \cong \mathcal{V}^B(M_{0,5})^\oplus 3 \)

\[ \Rightarrow \underline{\varpi} : \mathcal{V}^B(M_{0,5}) \to M_3(\mathcal{V}^B(M_{0,5})) \]

\( \underline{\varpi} \) is an algebra morphism.

row
\[ \text{row}_1 := \left( (X_1 - 1) \otimes 1, \ 1 \otimes (1 - X_1), \ 0 \right) \in M_{1 \times 3}((\mathcal{V}^B)^\otimes 2) \]

col
\[ \text{col}_1 := \begin{pmatrix} 1 \otimes 1 \\ -1 \otimes 1 \\ 0 \end{pmatrix} \in M_{3 \times 1}((\mathcal{V}^B)^\otimes 2) \]

Define an algebra morphism

\[ \rho : \mathcal{V}^B \to M_3((\mathcal{V}^B)^\otimes 2) \]

as

\[ \mathcal{V}^B \xrightarrow{\ell} \mathcal{V}^B(M_{0,5}) \xrightarrow{\underline{\varpi}} M_3(\mathcal{V}^B(M_{0,5})) \xrightarrow{M_3(\text{pr}_{12})} M_3((\mathcal{V}^B)^\otimes 2). \]
Proposition: The following diagram commutes

\[
\begin{array}{ccc}
\mathcal{V}^B & \xrightarrow{\rho} & M_3((\mathcal{V}^B)^\otimes 2) \\
& & \xrightarrow{\text{row}_1 \cdot (-) \cdot \text{col}_1}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{W}^B & \xrightarrow{\Delta^\mathcal{W}} & (\mathcal{W}^B)^\otimes 2
\end{array}
\]

Proof. Similar to De Rham case.
Proof of "algebra" comparison result

Follows from the commutativity of the following diagram
where
\[ e_1 := e_1 \otimes 1, \quad f_1 := 1 \otimes e_1, \quad X_1 := X_1 \otimes 1, \quad Y_1 := 1 \otimes X_1, \]
\[ u := B_\Phi \cdot \frac{e^{\mu e_1} - 1}{e_1} \cdot \frac{1 - e^{-\mu f_1}}{f_1}, \quad v = u^{-1} \cdot \frac{e^{\mu(e_1 + f_1)} - 1}{e_1 + f_1}, \]
\[ \kappa(\mu, \Phi) := e^{-(\mu/2)f_1} \Phi(e_0, e_1) \Phi(f_0, f_1) \in ((\hat{\mathcal{V}}^{DR}) \hat{\otimes} 2)^{\times}, \]
\[ P(\mu, \Phi) \in GL_3((U \mathfrak{g}_5)^\wedge) \]
is defined by
\[ \text{comp}^{((\bullet)\bullet)\bullet} \begin{pmatrix} x_{15} - 1 \\ x_{25} - 1 \\ x_{35} - 1 \end{pmatrix} = P(\mu, \Phi) \cdot \begin{pmatrix} e_{15} \\ e_{25} \\ e_{35} \end{pmatrix} \]
and
\[ \overline{P}(\mu, \Phi) := \text{pr}_{12}(P(\mu, \Phi)) \in GL_3((\hat{\mathcal{V}}^{DR}) \hat{\otimes} 2). \]
Commutativity of big diagram implies that of

\[
\begin{array}{cccc}
(\hat{\mathcal{W}}^B)_+ & \xrightarrow{\hat{\Delta}_*} & \hat{\mathcal{W}}^B[\frac{1}{X_1-1}]^\otimes 2 \\
\text{comp}^{(1),\mathcal{W}}_{(\mu,\Phi)} & & (\text{comp}^{(1),\mathcal{W}}_{(\mu,\Phi)})^\otimes 2 \\
(\hat{\mathcal{W}}^\text{DR})_+ & \xrightarrow{\hat{\Delta}_*} & \hat{\mathcal{W}}^\text{DR}[\frac{1}{e_1}]^\otimes 2 \\
(-) \cdot \frac{e_1}{e^{\mu e_1-1}} & \downarrow & B_{\Phi}^{-1} \cdot (-) \cdot B_{\Phi} \frac{e_1+f_1}{e^{\mu(e_1+f_1)-1}} \\
(\hat{\mathcal{W}}^\text{DR})_+ & \xrightarrow{\hat{\Delta}_*} & \hat{\mathcal{W}}^\text{DR}[\frac{1}{e_1}]^\otimes 2 \\
\end{array}
\]

which by \[\Delta^{\mathcal{W}}_\star \left( \frac{e_1}{e^{\mu e_1-1}} \right) = \frac{e_1+f_1}{e^{\mu(e_1+f_1)-1}}\] implies comm. of "algebra" comparison diagram.
Comm. of S1, S6: geometric interpretations of \( \Delta^{W}_{\#}, \Delta^{W}_{\star} \).

Comm. of S2: algebra morphism nature of \( \text{comp}^{(1),V}_{(\mu,\Phi)} \), its compatibility with \( \text{comp}^{(1),W}_{(\mu,\Phi)} \), its property \( X_1 \mapsto e^{\mu e_1} \).

Comm. of S5: same properties of \( \text{comp}^{(1),V}_{(\mu,\Phi)} \), identities relating \( u, B_{\Phi}, e_1, f_1 \) and \( v, B_{\Phi}, e_1, f_1 \).

Comm. of S3: \( \rho \) (resp. \( \overline{\rho} \)) is based on choice of basis \((e_{i5})_{i=1,2,3}\) (resp. \((x_{i5} - 1)_{i=1,2,3}\)) for \( \ker(U\mathfrak{p}_5 \to V^{\text{DR}}) \) (resp. \( \ker(\mathbb{C}P_5 \to V^{\text{B}}) \)), and \( P_{(\mu,\Phi)} \) expresses comparison of these bases.
Comm. of S4. is a consequence of the equalities

\[(\text{comp}_{(\mu,\Phi)}^{(1),V})\otimes^2(\text{col}_1) = \kappa_{(\mu,\Phi)}\overline{P}_{(\mu,\Phi)} \cdot \text{col}_1 \cdot \nu,\]

\[(\text{comp}_{(\mu,\Phi)}^{(1),V})\otimes^2(\text{row}_1) = u \cdot \text{row}_1 \cdot (\kappa_{(\mu,\Phi)}\overline{P}_{(\mu,\Phi)})^{-1},\]

whose proofs necessitate explicit computation:

(a) one expresses the braid group elements \(x_{i5}\)

\[
x_{15} = \quad x_{25} = \quad x_{35} = \quad x_{45} =
\]

as products of \(\sigma_{a,b}\)

\[
\sigma_{a,b} = \quad \in B_{a+b}
\]
(b) this enables one to compute explicitly
\[ \text{comp}_{(\mu, \Phi)}^{(\bullet \bullet \bullet \bullet)} (x_{i5} - 1) \text{ as elements of } \]
\[ U(\mathfrak{f}_3)^\wedge = \mathbb{C} \langle \langle e_{i5}, i = 1, \ldots, 4 \rangle \rangle; \]
(c) one derives from there the computation of \( P_{(\mu, \Phi)} \);
(d) one further derives the computation of \( \overline{P}_{(\mu, \Phi)} \);
(e) one plugs the obtained value into the first identity;
(f) the second identity can be similarly obtained. \( \square \)
Section 4: Proof of "module" comparison result
Set

\[
\text{col}_0 := \begin{pmatrix}
0 \\
-e_1 \cdot 1_{\text{DR}}^\otimes 2 \\
e_1 \cdot 1_{\text{DR}}^\otimes 2
\end{pmatrix} \in M_{3 \times 1}((M_{\text{DR}}^\otimes)^2).
\]

**Proposition:** The following diagram commutes

\[
\begin{array}{ccc}
\mathcal{V}^\text{DR} & \overset{\rho}{\longrightarrow} & M_3((\mathcal{V}^\text{DR})^\otimes 2) \\
\text{can} \downarrow & & \downarrow (e_1 f_1)^{-1} \text{row}_1 \cdot (-) \cdot \text{col}_0 \\
M_{\text{DR}} & \overset{\Delta^M}{\longrightarrow} (M_{\text{DR}}^\otimes)^2 & \overset{}{\longrightarrow} (M_{\text{DR}}^\otimes [1/e_1])^\otimes 2
\end{array}
\]
Proof.

(a) Show that \( \rho(e_0) \cdot \text{col}_0 = 0 \).

(b) Derive the existence of map \( \delta : \mathcal{M}^{\text{DR}} \to (\mathcal{M}^{\text{DR}}[\frac{1}{e_1}])^{\otimes 2} \) such that

\[
\begin{array}{ccc}
\mathcal{V}^{\text{DR}} & \xrightarrow{\rho} & M_3((\mathcal{V}^{\text{DR}})^{\otimes 2}) \\
\downarrow \text{can} & & \downarrow (e_1 f_1)^{-1} \text{row}_1 \cdot (-) \cdot \text{col}_0 \\
\mathcal{M}^{\text{DR}} & \xrightarrow{\delta} & (\mathcal{M}^{\text{DR}}[\frac{1}{e_1}])^{\otimes 2}
\end{array}
\]

commutes.

(c) Using \( \rho(e_1) = \text{col}_1 \cdot \text{row}_1 \), show that \( \delta \) is compatible with \( \Delta^\mathcal{W} : \mathcal{W}^{\text{DR}} \to (\mathcal{W}^{\text{DR}})^{\otimes 2} \) and module structure of \( \mathcal{M}^{\text{DR}} \) over \( \mathcal{W}^{\text{DR}} \).

(d) Compute \( (e_1 f_1)^{-1} \text{row}_1 \cdot \text{col}_0 = 1_{\text{DR}}^{\otimes 2} \) to get \( \delta(1_{\text{DR}}) = 1_{\text{DR}}^{\otimes 2} \).

(e) Derive \( \delta = \Delta^\mathcal{M} \).
Interpretation of $\Delta^M_#$ in terms of moduli spaces $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$

Set

$$\text{col}_0 := \begin{pmatrix} 0 \\ (1 - X_1)Y_1^{-1} \cdot 1^B \otimes^2 \\ (1 - X_1^{-1})Y_1^{-1} \cdot 1^B \otimes^2 \end{pmatrix} \in M_{3 \times 1}((\mathcal{M}^B \otimes^2)).$$

Proposition: The following diagram commutes

$$\begin{array}{ccc}
\mathcal{M}^B & \xrightarrow{\Delta^M_#} & (\mathcal{M}^B \otimes^2) \\
\downarrow \text{can} & & \downarrow \text{row} \\
\mathcal{V}^B & \xrightarrow{\rho} & M_3((\mathcal{V}^B \otimes^2))
\end{array}$$

Proof. Similar to De Rham case. \qed
Proof of the “module” comparison result

Follows from the commutativity of the following diagram

where \( Q_{(\mu, \Phi)} \in \text{GL}_3((\hat{\varnothing}^{\text{DR}}) \hat{\otimes}^2) \) is given by

\[
Q_{(\mu, \Phi)}^{-1} = \Phi(e_1, e_0) \Phi(f_1, f_0) \cdot \kappa_{(\mu, \Phi)} \overline{P}_{\mu, \Phi} \cdot \hat{\rho}(\Phi).
\]
Comm. of T1 and T5: by the above geometric interpretations of $\Delta^M$, $\Delta^M_\star$, $\Delta^M_\#$.

Comm. of T2: by construction.

Comm. of T3: states equality of two maps $\hat{\mathcal{V}}^B \rightarrow M_3((\hat{\mathcal{V}}^\text{DR})\hat{\otimes}^2)$.

These maps are module morphisms over two algebra morphisms $\hat{\mathcal{V}}^B \rightarrow M_3((\hat{\mathcal{V}}^\text{DR})\hat{\otimes}^2)$ which are two parts of S3 and turn out to be equal due to comm. of S3.

The value taken by $Q_{(\mu,\Phi)}$ guarantees that these maps agree on generator $1 \in \mathcal{V}^\text{DR}$. 

Comm. of T4 is a consequence of the equalities

\[
\left(\text{comp}_{(\mu,\Phi)}^{(1),\mathcal{V}}\right) \otimes^2 ((X_1 - 1)^{-1}(1 - Y_1^{-1})^{-1}\text{row}_1) = \Gamma_{\Phi}(-e_1)\Gamma_{\Phi}(-f_1)
\]
\[
= \frac{\Gamma_{\Phi}(-e_1)}{\Gamma_{\Phi}(-e_1 - f_1)} (e_1 f_1)^{-1} \cdot \text{row}_1 \cdot (\kappa_{(\mu,\Phi)} \bar{P}_{(\mu,\Phi)})^{-1}
\]

and

\[
\left(\text{comp}_{(\mu,\Phi)}^{(10),\mathcal{M}}\right) \otimes^2 \text{col}_0 = \left(\text{comp}_{(\mu,\Phi)}^{(10),\mathcal{V}}\right) \otimes^2 (1) \cdot \kappa_{(\mu,\Phi)} Q_{(\mu,\Phi)}^{-1} \cdot \text{col}_0.
\]

First equality: an immediate consequence of already proved equality (proof of S4).
Second equality: \textit{explicit computation} parallel to previous one but dealing with \((\bullet(\bullet\bullet))\bullet\) rather than \(((\bullet\bullet))\bullet\):
Proof of the “module” comparison result

(a) one expresses the braid group elements $x_{i5}$ as products of $\sigma_{a,b}$ (same expressions as before except for $x_{25}$);
(b) this enables one to compute explicitly $\text{comp}^{\bullet(\bullet)}(x_{i5} - 1)$ as elements of $U(\hat{f}_3)^\wedge = \mathbb{C}\langle\langle e_{i5}, i = 1, \ldots, 4\rangle\rangle$;
(c) one derives the computation of $R_{(\mu, \Phi)} \in \text{GL}_3((U \hat{p}_5)^\wedge)$ defined by

$$\text{comp}^{\bullet(\bullet)}\begin{pmatrix} x_{15} - 1 \\ x_{25} - 1 \\ x_{35} - 1 \end{pmatrix} = R_{(\mu, \Phi)} \cdot \begin{pmatrix} e_{15} \\ e_{25} \\ e_{35} \end{pmatrix}$$

(d) and therefore of $R_{(\mu, \Phi)} := \text{pr}_{12}(R_{(\mu, \Phi)}) \in \text{GL}_3((\hat{V}^{\text{DR}})\hat{\otimes}^2)$.
(e) one proves that $Q^{-1}_{(\mu, \Phi)} = \Phi(e_1, e_0)\Phi(f_1, f_0)\kappa_{(\mu, \Phi)} R_{(\mu, \Phi)}$
(f) one then computes $Q^{-1}_{(\mu, \Phi)}$.
(g) one uses the obtained expression of $Q^{-1}_{(\mu, \Phi)}$ to explicitly prove wanted equality. □