On double shuffle relations for MZVs

collaboration with H. Furusho (Nagoya)

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Papers

- *The Betti side of the double shuffle theory. I. The harmonic coproduct*, `arXiv:1803.10151`.
- *The Betti side of the double shuffle theory. II. Torsor structures*, `arXiv:1807.07786`.
- *The Betti side of the double shuffle theory. III. Double shuffle relations for associators*, in preparation.

Contents

Section 0:
The context

Two approaches to the algebraic relations between MZVs

- based on combinatorics: The MZVs satisfy double shuffle relations (Ihara-Kaneko-Zagier 2006, Racinet 2002).
- based on the geometry of moduli space of curves: The "KZ associator" (Drinfeld) is a generating series for MZVs (Le-Murakami 1996). It satisfies algebraic relations (Drinfeld 1991).

Relations between the two approaches

Thm (Furusho 2011, Deligne-Terasoma 2005 (announcement)).
The associator relations imply the double shuffle relations.

- Ideas of (Furusho 2011): Associator relations take place in $U\mathfrak{p}_5$. Construction of explicit linear forms on $U\mathfrak{p}_5$, based on multiple polylogs. Combinatorics of linear forms.
- Ideas of (Deligne-Terasoma 2005): Geometric constructions with moduli spaces $\mathfrak{M}_{0,4}$ and $\mathfrak{M}_{0,5}$. Perverse sheaves on these spaces. Redaction is still unfinished.

Remark: [Hirose-Sato 2018+] and [Furusho 2018+] give another proof.

Today: New proof of theorem based on Deligne-Terasoma ideas.

Detailed plan: (1)

(1) The double shuffle formalism

(1a) MZVs

(1b) Examples of double shuffle relations

(1c) The double shuffle formalism:

- algebra $\mathcal{W}^{\mathbf{DR}}$ and coproduct $\Delta_{\star}^{\mathcal{W}}$ on it (harmonic coproduct)
- a rank 1 module $\mathcal{M}^{\mathbf{DR}}$ over it and a coproduct $\Delta_{\star}^{\mathcal{M}}$ over this module
- Γ-functions $\Gamma_{\Phi}(t)$

(1d) Formulation of double shuffle relation in terms of double shuffle formalism.

Detailed plan: (2)

(2) Two comparison results and the main result

(2a) Betti version $(\mathcal{W}^{\mathbf{B}}, \Delta_\sharp^{\mathcal{W}}, \mathcal{M}^{\mathbf{B}}, \Delta_\sharp^{\mathcal{M}})$ of $(\mathcal{W}^{\mathbf{DR}}, \Delta_\star^{\mathcal{W}}, \mathcal{M}^{\mathbf{DR}}, \Delta_\star^{\mathcal{M}})$

(2b) "comparison" operators $\mathbf{comp}_{(\mu,\Phi)}^{(1),\mathcal{W}} : \mathcal{W}^{\mathbf{DR}} \to \mathcal{W}^{\mathbf{B}}$ and $\mathbf{comp}_{(\mu,\Phi)}^{(10),\mathcal{M}} : \mathcal{M}^{\mathbf{DR}} \to \mathcal{M}^{\mathbf{B}}$.

(2c) comparison results:
(algebra): for (μ, Φ) associator, $\mathbf{comp}_{(\mu,\Phi)}^{(1),\mathcal{W}}$ brings $\Delta_\star^{\mathcal{W}}$ to $\Delta_\sharp^{\mathcal{W}}$
(module): for (μ, Φ) associator, $\mathbf{comp}_{(\mu,\Phi)}^{(10),\mathcal{M}}$ brings $\Delta_\star^{\mathcal{M}}$ to $\Delta_\sharp^{\mathcal{M}}$

(2d) why the "module" comparison result implies the associator-double shuffle implication (main result).

Detailed plan: (3)

(3) Proof of "algebra" comparison result

(3a) Interpretation of $\Delta_\star^{\mathcal{W}}$ in terms of moduli spaces $\mathfrak{M}_{0,4}$ and $\mathfrak{M}_{0,5}$ (Deligne-Terasoma)

(3b) Interpretation of $\Delta_\sharp^{\mathcal{W}}$ in terms of moduli spaces $\mathfrak{M}_{0,4}$ and $\mathfrak{M}_{0,5}$

(3c) Proof of comparison result based on study of $\mathbf{comp}_{(\mu,\Phi)} : \mathbf{PaB} \to \widehat{\mathbf{PaCD}}$ evaluated at $((\bullet\bullet)\bullet)\bullet$

Detailed plan: (4)

(4) Proof of "module" comparison result

(4a) Interpretation of $\Delta_{\star}^{\mathcal{M}}$ in terms of moduli spaces $\mathfrak{M}_{0,4}$ and $\mathfrak{M}_{0,5}$

(4b) Interpretation of $\Delta_{\sharp}^{\mathcal{M}}$ in terms of moduli spaces $\mathfrak{M}_{0,4}$ and $\mathfrak{M}_{0,5}$

(4c) Proof of comparison result based on study of $\mathbf{comp}_{(\mu,\Phi)} : \mathbf{PaB} \to \widehat{\mathbf{PaCD}}$ evaluated at $(\bullet(\bullet\bullet))\bullet$

Section 1: The double shuffle formalism

Multiple Zeta Value (MZV)

For $k_1, \ldots, k_{m-1} \geqslant 1$ and $k_m \geqslant 2$,

$$\zeta(k_1, \ldots, k_m) := \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}} \in \mathbb{R} : \text{MZV}$$

- The sum converges iff $k_m > 1$.
- $m = 1$: Riemann zeta value $\zeta(k)$.
- $m = 2$: Double zeta value by Goldbach and Euler.

Double Shuffle relations for MZV's
'=' Shuffle + Harmonic product

Shuffle product:

e.g.

$$\zeta(a)\zeta(b) = \int_{0<s_1<\cdots<s_a<1} \frac{ds_1}{1-s_1} \wedge \frac{ds_2}{s_2} \wedge \cdots \wedge \frac{ds_a}{s_a}$$

$$\times \int_{0<t_1<\cdots<t_b<1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2} \wedge \cdots \wedge \frac{dt_b}{t_b}$$

$$= \sum \int_0^1 \text{all shuffles}$$

$$= \sum_{i+j=a+b} \left\{ \binom{i-1}{a-1} + \binom{j-1}{b-1} \right\} \zeta(i,j)$$

Harmonic product:

e.g.

$$\zeta(a)\zeta(b) = \sum_{0<k} \frac{1}{k^a} \cdot \sum_{0<l} \frac{1}{l^b} = \Big(\sum_{0<k<l} + \sum_{0<k=l} + \sum_{0<l<k} \Big) \frac{1}{k^a l^b}$$

$$= \zeta(a,b) + \zeta(a+b) + \zeta(b,a).$$

The double shuffle formalism (Racinet)

- $\mathcal{V}^{\mathrm{DR}} := \mathbb{C}\langle e_0, e_1\rangle$:free graded algebra over e_0, e_1 of deg=1.
  Coproduct $\Delta : \mathcal{V}^{\mathrm{DR}} \to (\mathcal{V}^{\mathrm{DR}})^{\otimes 2}$, $e_i \mapsto e_i \otimes 1 + 1 \otimes e_i$.
- Subalgebra $\mathcal{W}^{\mathrm{DR}} := \mathbb{C} \oplus \mathcal{V}^{\mathrm{DR}} e_1 (\hookrightarrow \mathcal{V}^{\mathrm{DR}})$.
  Presentation: $\mathcal{W}^{\mathrm{DR}}$ is freely generated by $y_1, y_2, \ldots$, where
  $y_n := -e_0^{n-1} e_1$.
  Harmonic coproduct $\Delta_\star^{\mathcal{W}} : \mathcal{W}^{\mathrm{DR}} \to (\mathcal{W}^{\mathrm{DR}})^{\otimes 2}$,
  $\Delta_\star^{\mathcal{W}}(y_n) = y_n \otimes 1 + 1 \otimes y_n + \sum_{k+l=n} y_k \otimes y_l$
  equips $\mathcal{W}^{\mathrm{DR}}$ with Hopf algebra structure.

- Quotient $\mathcal{M}^{\mathbf{DR}} := \mathcal{V}^{\mathbf{DR}}/\mathcal{V}^{\mathbf{DR}}e_0$and the canonical projection can : $\mathcal{V}^{\mathbf{DR}} \twoheadrightarrow \mathcal{M}^{\mathbf{DR}}$. Then $\mathcal{M}^{\mathbf{DR}}$ is a free $\mathcal{W}^{\mathbf{DR}}$-module of rank 1, generated by $\mathbf{1}_{\mathbf{DR}}$:=projection of $\mathbf{1} \in \mathcal{V}^{\mathbf{DR}}$. Define $\Delta_\star^{\mathcal{M}} : \mathcal{M}^{\mathbf{DR}} \to (\mathcal{M}^{\mathbf{DR}})^{\otimes 2}$ as the transport of $\Delta_\star^{\mathcal{W}}$ under the isomorphism $\mathcal{W}_{\mathbf{DR}} \to \mathcal{M}_{\mathbf{DR}}$ induced by action on $\mathbf{1}_{\mathbf{DR}}$.

Notation:

For $\Phi \in \hat{\mathcal{V}}^{\mathbf{DR}} := \mathbb{C}\langle\langle e_0, e_1\rangle\rangle$, set

$$\Gamma_\Phi(-e_1)^{-1} := \exp\Big(\sum_{n\geq 1} \frac{1}{n}(\Phi|e_0^{n-1}e_1)e_1^n\Big) \in \hat{\mathcal{V}}^{\mathbf{DR}},$$

$$\Phi_\star := \mathrm{can}(\Gamma_\Phi(-e_1)^{-1}\Phi) \in \hat{\mathcal{M}}^{\mathbf{DR}}.$$

Generating series of MZVs:

$$\Phi_{\mathrm{KZ}} := 1 + \sum (-1)^m \zeta(k_1, \ldots, k_m) \cdot e_0^{k_m-1} e_1 \cdots e_0^{k_1-1} e_1$$
$$+ \text{(terms in } e_1 \mathcal{V}^{\mathrm{DR}} + \mathcal{V}^{\mathrm{DR}} e_0) \in \hat{\mathcal{V}}^{\mathrm{DR}}.$$

Relations:

shuffle relation: $\hat{\Delta}(\Phi_{\mathrm{KZ}}) = \Phi_{\mathrm{KZ}} \otimes \Phi_{\mathrm{KZ}}$ (relation in $(\hat{\mathcal{V}}^{\mathrm{DR}})^{\hat{\otimes} 2}$)

harmonic relation: $\hat{\Delta}_{\star}^{\mathcal{M}}(\Phi_{\mathrm{KZ},\star}) = \Phi_{\mathrm{KZ},\star} \otimes \Phi_{\mathrm{KZ},\star}$ (relation in $(\hat{\mathcal{M}}^{\mathrm{DR}})^{\hat{\otimes} 2}$).

One says that the collection of commutative variables $\zeta^f(k_1, \ldots, k_m)$ satisfy the double shuffle relations iff they satisfy the above relations with Φ_{KZ} replaced by

$$\Phi := 1 + \sum (-1)^m \zeta^f(k_1, \ldots, k_m)(e_0^{k_m-1} e_1 \cdots e_0^{k_1-1} e_1$$
$$+ \text{(terms in } e_1 \mathcal{V}^{\mathrm{DR}} + \mathcal{V}^{\mathrm{DR}} e_0).$$

Section 2:
The comparison results and the main result

Betti version $(\mathcal{W}^{\mathrm{B}}, \Delta_\sharp^{\mathcal{W}}, \mathcal{M}^{\mathrm{B}}, \Delta_\sharp^{\mathcal{M}})$ of $(\mathcal{W}^{\mathrm{DR}}, \Delta_\star^{\mathcal{W}}, \mathcal{M}^{\mathrm{DR}}, \Delta_\star^{\mathcal{M}})$

- Algebra $\mathcal{V}^{\mathrm{B}} := \mathbb{C}F_2$, where $F_2 :=$ free group over X_0, X_1. Coproduct $\underline{\Delta}$: X_0, X_1 are group-like.

- Subalgebra $\mathcal{W}^{\mathrm{B}} := \mathbb{C} \oplus \mathcal{V}^{\mathrm{B}}(X_1 - 1)$ $(\hookrightarrow \mathcal{V}^{\mathrm{B}})$.
  Presentation: generators
  $X_1, X_1^{-1}, Y_n^{\pm} := (X_0^{\pm 1} - 1)^{n-1} X_0^{\pm 1}(1 - X_1^{\pm 1})$ $(n \geq 1)$
  with only relations $X_1 \cdot X_1^{-1} = X_1^{-1} \cdot X_1 = 1$.
  Coproduct on $\mathcal{W}^{\mathrm{B}}$ is $\Delta_\sharp^{\mathcal{W}} : \mathcal{W}^{\mathrm{B}} \to (\mathcal{W}^{\mathrm{B}})^{\otimes 2}$ given by

$$\Delta_\sharp^{\mathcal{W}}(X_1^{\pm 1}) = X_1^{\pm 1} \otimes X_1^{\pm 1},$$

$$\Delta_\sharp^{\mathcal{W}}(Y_n^{\pm}) = Y_n^{\pm} \otimes 1 + 1 \otimes Y_n^{\pm} + \sum_{k+l=n} Y_k^{\pm} \otimes Y_l^{\pm}$$

  equips $\mathcal{W}^{\mathrm{B}}$ with a Hopf algebra structure.

- Quotient vector space $\mathcal{M}^{\mathbf{B}} := \mathcal{V}^{\mathbf{B}}/\mathcal{V}^{\mathbf{B}}(X_0 - \mathbf{1})$. Set $\mathbf{1_B}$:=projection of $\mathbf{1} \in \mathcal{V}^{\mathbf{B}}$. Then $\mathcal{M}^{\mathbf{B}}$ is a free $\mathcal{W}^{\mathbf{B}}$-module generated by $\mathbf{1_B}$. Define

$$\Delta_\sharp^{\mathcal{M}} : \mathcal{M}^{\mathbf{B}} \to (\mathcal{M}^{\mathbf{B}})^{\otimes 2}$$

as the transport of $\Delta_\sharp^{\mathcal{W}} : \mathcal{W}^{\mathbf{B}} \to (\mathcal{W}^{\mathbf{B}})^{\otimes 2}$ under the isomorphism $\mathcal{W}^{\mathbf{B}} \to \mathcal{M}^{\mathbf{B}}$ induced by action on $\mathbf{1_B}$.

Comparison operator $\mathbf{comp}^{(1),\mathcal{W}}_{(\mu,\Phi)} : \mathcal{W}^{\mathrm{DR}} \to \mathcal{W}^{\mathrm{B}}$

Algebra isomorphisms: For $(\mu, \Phi) \in \mathbb{C}^\times \times \mathcal{G}(\hat{\mathcal{V}}^{\mathrm{DR}})$ (notation: $\mathcal{G}(-)$=group of group-like elements of a Hopf algebra), define algebra isomorphism

$$\mathbf{comp}^{(1),\mathcal{V}}_{(\mu,\Phi)} : \hat{\mathcal{V}}^{\mathrm{B}} \to \hat{\mathcal{V}}^{\mathrm{DR}},$$

$$X_0 \mapsto \Phi \cdot \exp(\mu e_0) \cdot \Phi^{-1}, \quad X_1 \mapsto \exp(\mu e_1).$$

Note: when $(\mu, \Phi) = (2\pi i, \Phi_{\mathrm{KZ}})$, this is the period isomorphism $\mathbb{C}\pi_1^{\mathrm{B}}(\mathfrak{M}_{0,4}, \vec{1})^\wedge \to \mathbb{C}\pi_1^{\mathrm{DR}}(\mathfrak{M}_{0,4}, \vec{1})^\wedge$ between the Betti and De Rham fundamental group algebras.

The isomorphism $\mathbf{comp}^{(1),\mathcal{V}}_{(\mu,\Phi)}$ restricts to an algebra isomorphism

$$\mathbf{comp}^{(1),\mathcal{W}}_{(\mu,\Phi)} : \hat{\mathcal{W}}^{\mathrm{B}} \to \hat{\mathcal{W}}^{\mathrm{DR}}.$$

Comparison operator $\mathbf{comp}^{(10),\mathcal{M}}_{(\mu,\Phi)} : \mathcal{M}^{DR} \to \mathcal{M}^{B}$

Module isomorphisms: For (μ, g) in $\mathbb{C}^\times \times \mathcal{G}(\hat{\mathcal{V}}^{DR})$, define

$$\mathbf{comp}^{(10),\mathcal{V}}_{(\mu,\Phi)} : \hat{\mathcal{V}}^{B} \to \hat{\mathcal{V}}^{DR}, \quad v \mapsto \mathbf{comp}^{(1),\mathcal{V}}_{(\mu,\Phi)}(v) \cdot \Phi.$$

Note: when $(\mu, \Phi) = (2\pi i, \Phi_{KZ})$, this is the period isomorphism $\mathbb{C}\pi_1^{B}(\mathfrak{M}_{0,4}, \vec{1}, \vec{0})^\wedge \to \mathbb{C}\pi_1^{DR}(\mathfrak{M}_{0,4}, \vec{1}, \vec{0})^\wedge$ between the Betti and De Rham fundamental groupoid modules.

This isomorphism factors to an isomorphism

$$\mathbf{comp}^{\mathcal{M}}_{(\mu,\Phi)} : \hat{\mathcal{M}}^{B} \to \hat{\mathcal{M}}^{DR}.$$

Summary

We summarize the situation as follows:

	algebras		modules over $\mathcal{V}^{\mathbf{B/DR}}$	
morphisms	$\mathcal{W}^{\mathbf{B/DR}} \hookrightarrow \mathcal{V}^{\mathbf{B/DR}}$		$\mathcal{V}^{\mathbf{B/DR}} \twoheadrightarrow \mathcal{M}^{\mathbf{B/DR}}$	
coproduct	$\Delta_\sharp/\Delta_\star$	Δ/Δ		$\Delta^{\mathcal{M}}_\sharp/\Delta^{\mathcal{M}}_\star$
fake B/DR isoms	$\mathbf{comp}^{\mathcal{W}}_{(\mu,\Phi)}$	$\mathbf{comp}^{(1),\mathcal{V}}_{(\mu,\Phi)}$	$\mathbf{comp}^{(10),\mathcal{V}}_{(\mu,\Phi)}$	$\mathbf{comp}^{\mathcal{M}}_{(\mu,\Phi)}$
geometry	$\pi_1(\mathfrak{M}_{0,4}; \vec{1})$		$\pi_1(\mathfrak{M}_{0,4}; \vec{1}, \vec{0})$	

Associators and comparison isomorphisms

Definition: (Drinfeld 1991, Furusho 2010)
An associator is a pair $(\mu, \Phi) \in \tilde{G}^{\mathrm{DR}} = \mathbb{C}^\times \times (\hat{\mathcal{V}}^{\mathrm{DR}})^\times$ *(recall that* $(\hat{\mathcal{V}}^{\mathrm{DR}})^\times = \mathbb{C}\langle\langle e_0, e_1\rangle\rangle^\times$*) such that*

- $(\Phi|e_0) = (\Phi|e_1) = 0$,
- $(\Phi|e_0e_1) = \mu^2/24$,
- $\hat{\Delta}(\Phi) = \Phi \otimes \Phi$,
- $\Phi^{345}\Phi^{512}\Phi^{234}\Phi^{451}\Phi^{123} = 1$ *in* $(U\mathfrak{P}_5)^\wedge$.

Example: $(\mu, \Phi) = (2\pi i, \Phi_{\mathrm{KZ}})$ is an associator.

Main property of associators (Drinfeld 1991, Bar-Natan 1998)

An associator (μ, Φ) gives rise to a functor

$$\mathbf{comp}_{(\mu,\Phi)} : \mathbf{PaB} \to \widehat{\mathbf{PaCD}}$$

between the categories of parenthesized braids and parenthasized chord diagrams.

Specializing this functor to sets of morphisms, one gets a system of isomorphisms of topological vector spaces

$$\mathbf{comp}^{(n),\vec{a},\vec{b}}_{(\mu,\Phi)} : \mathbb{C}\pi_1^{\mathrm{B}}(\mathfrak{M}_{0,n}; \vec{a}, \vec{b}) \to \mathbb{C}\pi_1^{\mathrm{DR}}(\mathfrak{M}_{0,n}; \vec{a}, \vec{b})$$

where $\vec{a}, \vec{b}$ are tangential base points of $\mathfrak{M}_{0,n}$.

Particular cases: $\mathbf{comp}^{\mathcal{V},(1)}_{(\mu,\Phi)}$ $(n = 4, (\vec{a}, \vec{b}) = (\vec{1}, \vec{1}))$, $\mathbf{comp}^{\mathcal{V},(10)}_{(\mu,\Phi)}$ $(n = 4, (\vec{a}, \vec{b}) = (\vec{1}, \vec{0}))$.

"Algebra" comparison result

If (μ, Φ) is an associator, then the following diagram commutes

$$\begin{array}{ccccc}
\hat{\mathcal{W}}^{\mathrm{B}} & & \xrightarrow{\quad \underline{\hat{\Delta}}_{\sharp}^{\mathcal{W}} \quad} & & (\hat{\mathcal{W}}^{\mathrm{B}})^{\hat{\otimes}2} \\
\mathrm{comp}^{\mathcal{W}}_{(\mu,\Phi)} \downarrow \simeq & & & & \simeq \downarrow (\mathrm{comp}^{\mathcal{W}}_{(\mu,\Phi)})^{\hat{\otimes}2} \\
\hat{\mathcal{W}}^{\mathrm{DR}} & \xrightarrow[\hat{\Delta}_{\star}^{\mathcal{W}}]{} & (\hat{\mathcal{W}}^{\mathrm{DR}})^{\hat{\otimes}2} & \xrightarrow[\mathrm{Ad}(B_\Phi)]{\simeq} & (\hat{\mathcal{W}}^{\mathrm{DR}})^{\hat{\otimes}2}
\end{array}$$

Here

$$B_\Phi := \frac{\Gamma_\Phi(-e_1 \otimes 1)\Gamma_\Phi(-1 \otimes e_1)}{\Gamma_\Phi(-e_1 \otimes 1 - 1 \otimes e_1)} \in ((\hat{\mathcal{W}}^{\mathrm{DR}})^{\hat{\otimes}2})^{\times}.$$

The proof of this result will be an ingredient in the proof of the next result:

"Module" comparison result

If (μ, Φ) is an associator, then the following diagram commutes

$$
\begin{array}{ccccc}
\hat{\mathcal{M}}^{\mathbf{B}} & & \xrightarrow{\quad \hat{\Delta}^{\mathcal{M}}_{\sharp} \quad} & & (\hat{\mathcal{M}}^{\mathbf{B}})^{\hat{\otimes} 2} \\
\mathbf{comp}^{\mathcal{M}}_{(\mu,\Phi)} \downarrow \simeq & & & & \simeq \downarrow (\mathbf{comp}^{\mathcal{M}}_{(\mu,\Phi)})^{\hat{\otimes} 2} \\
\hat{\mathcal{M}}^{\mathbf{DR}} & \xrightarrow[\hat{\Delta}^{\mathcal{M}}_{\star}]{} & (\hat{\mathcal{M}}^{\mathbf{DR}})^{\hat{\otimes} 2} & \xrightarrow[B_{\Phi} \cdot (-)]{\simeq} & (\hat{\mathcal{M}}^{\mathbf{DR}})^{\hat{\otimes} 2}
\end{array}
$$

Why the "module" comparison result implies the associator-double shuffle implication (main result)?

Apply to $\mathbf{1_B} \in \hat{\mathcal{M}}^{\mathbf{B}}$: $\hat{\Delta}^{\mathcal{M}}_{\sharp}(\mathbf{1_B}) = \mathbf{1_B}^{\otimes 2}$.

Then $\mathbf{comp}^{\mathcal{M}}_{(\mu,\Phi)}(\mathbf{1_B}) = \Phi \cdot \mathbf{1_{DR}} = \mathbf{can}(\Phi)$.

So $\Delta^{\mathcal{M}}_{\star}(\mathbf{can}(\Phi)) = B_{\Phi}^{-1} \cdot \mathbf{can}(\Phi)^{\otimes 2}$. Hence $\Delta^{\mathcal{M}}_{\star}(\Phi_{\star}) = \Phi_{\star}^{\otimes 2}$. □

Section 3:
Proof of ”algebra” comparison result

Interpretation of $\Delta_\star^{\mathcal{W}}$ in terms of moduli spaces $\mathfrak{M}_{0,4}$ and $\mathfrak{M}_{0,5}$ (Deligne-Terasoma)

Recall that

$$\mathcal{V}_{\mathrm{DR}} \simeq U\mathrm{Lie}\pi_1^{\mathrm{DR}}(\mathfrak{M}_{0,4}; t_1).$$

Set:

- $$\mathcal{V}^{\mathrm{DR}}(\mathfrak{M}_{0,5}) := \mathrm{U}\mathfrak{P}_5 \simeq U\mathrm{Lie}\pi_1^{\mathrm{DR}}(\mathfrak{M}_{0,5}; t_{11}) \overset{\ell}{\hookleftarrow} \mathcal{V}^{\mathrm{DR}}$$
  $$e_{23}, e_{12} \leftarrow\!\shortmid e_0, e_1$$
  ℓ=algebra morphism
- $\mathrm{pr}_i : \mathfrak{M}_{0,5} \twoheadrightarrow \mathfrak{M}_{0,4} \qquad (i = 1, \ldots, 5)$
  $\Rightarrow \mathrm{pr}_i : \mathcal{V}^{\mathrm{DR}}(\mathfrak{M}_{0,5}) \twoheadrightarrow \mathcal{V}_{\mathrm{DR}}$
  pr_i are algebra morphisms and $\mathrm{pr}_5 \circ \ell = \mathrm{id}$
  $\mathrm{pr}_{12} : \mathcal{V}^{\mathrm{DR}}(\mathfrak{M}_{0,5}) \to (\mathcal{V}_{\mathrm{DR}})^{\otimes 2}$ defined by
  $\mathrm{pr}_{12} := (\mathrm{pr}_1 \otimes \mathrm{pr}_2) \circ \Delta.$

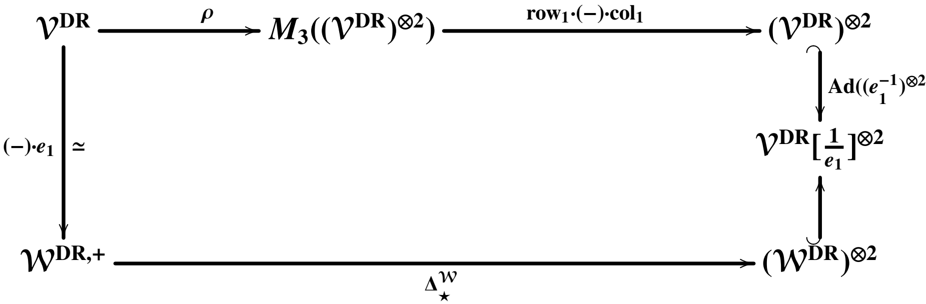
- $\mathcal{V}^{\mathrm{DR}}(\mathfrak{M}_{0,5}) \curvearrowright \ker\{\mathcal{V}^{\mathrm{DR}}(\mathfrak{M}_{0,5}) \stackrel{\mathrm{pr}_5}{\twoheadrightarrow} \mathcal{V}^{\mathrm{DR}}\} \simeq \mathcal{V}^{\mathrm{DR}}(\mathfrak{M}_{0,5})^{\oplus 3}$
  $\Rightarrow \varpi : \mathcal{V}^{\mathrm{DR}}(\mathfrak{M}_{0,5}) \to M_3(\mathcal{V}^{\mathrm{DR}}(\mathfrak{M}_{0,5}))$
  ϖ is an algebra morphim.
- $\mathbf{row}_1 := \begin{pmatrix} e_1 \otimes 1, & -1 \otimes e_1, & 0 \end{pmatrix} \in M_{1\times 3}((\mathcal{V}^{\mathrm{DR}})^{\otimes 2})$
- $\mathbf{col}_1 := \begin{pmatrix} 1 \otimes 1 \\ -1 \otimes 1 \\ 0 \end{pmatrix} \in M_{3\times 1}((\mathcal{V}^{\mathrm{DR}})^{\otimes 2})$

Define an algebra morphism

$$\rho : \mathcal{V}^{\mathrm{DR}} \to M_3((\mathcal{V}^{\mathrm{DR}})^{\otimes 2})$$

as $\mathcal{V}^{\mathrm{DR}} \stackrel{\ell}{\to} \mathcal{V}^{\mathrm{DR}}(\mathfrak{M}_{0,5}) \stackrel{\varpi}{\to} M_3(\mathcal{V}^{\mathrm{DR}}(\mathfrak{M}_{0,5})) \xrightarrow{M_3(\mathrm{pr}_{12})} M_3((\mathcal{V}^{\mathrm{DR}})^{\otimes 2})$.

Proposition: The following diagram commutes



Proof. (a) Define algebra $(\mathcal{V}^{\mathrm{DR}}, \cdot_{e_1})$ by $a \cdot_{e_1} b := a \cdot e_1 \cdot b$.
(b) Show that $\rho(e_1) = \mathbf{col}_1 \cdot \mathbf{row}_1$ and derive that $(\mathcal{V}^{\mathrm{DR}}, \cdot_{e_1}) \xrightarrow{\rho} M_3((\mathcal{V}^{\mathrm{DR}})^{\otimes 2}) \xrightarrow{\mathbf{row}_1 \cdot (-) \cdot \mathbf{col}_1} (\mathcal{V}^{\mathrm{DR}})^{\otimes 2}$ is an algebra morphism.
(c) The map $(\mathcal{V}^{\mathrm{DR}}, \cdot_{e_1}) \xrightarrow{(-)\cdot e_1} \mathcal{W}_+^{\mathrm{DR}}$ is also an algebra morphism.
(d) So if $\mathcal{V}^{\mathrm{DR}}$ is equipped with $\cdot_{e_1}$, all maps in diagram are algebra morphisms.
(d) Prove commutativity on each e_0^n by direct computation.
(e) Conclude from fact that $(\mathcal{V}^{\mathrm{DR}}, \cdot_{e_1})$ is algebra-generated by the e_0^n, $n \geq 0$. □

Interpretation of $\Delta_\sharp^{\mathcal{W}}$ in terms of moduli spaces $\mathfrak{M}_{0,4}$ and $\mathfrak{M}_{0,5}$

Recall that $\mathcal{V}^{\mathrm{B}} = \mathbb{C}\langle X_0^{\pm 1}, X_1^{\pm 1}\rangle = \mathbb{C}F_2 \simeq \mathbb{C}\pi_1^{\mathrm{topo}}(\mathfrak{M}_{0,4}; \vec{1})$,

$$\mathcal{W}^{\mathrm{B}} = \mathbb{C} \oplus \mathcal{V}^{\mathrm{B}} \cdot (X_1 - 1).$$

Set:

- $\mathcal{V}^{\mathrm{B}}(\mathfrak{M}_{0,5}) := \mathbb{C}P_5^* \simeq \mathbb{C}\pi_1^{\mathrm{topo}}(\mathfrak{M}_{0,5}; t_{11}) \overset{\underline{\ell}}{\hookleftarrow} \mathcal{V}^{\mathrm{B}}$
  $$x_{23}, x_{12} \leftarrow X_0, X_1$$
  $\underline{\ell}$ is an algebra morphism.
- $\mathrm{pr}_i : \mathfrak{M}_{0,5} \twoheadrightarrow \mathfrak{M}_{0,4} \qquad (i = 1, \ldots, 5)$
  $\Rightarrow \underline{\mathrm{pr}}_i : \mathcal{V}^{\mathrm{B}}(\mathfrak{M}_{0,5}) \twoheadrightarrow \mathcal{V}_{\mathrm{B}}$
  $\underline{\mathrm{pr}}_i$ is an algebra morphism and $\underline{\mathrm{pr}}_5 \circ \underline{\ell} = \mathrm{id}$
  $\underline{\mathrm{pr}}_{12} := (\underline{\mathrm{pr}}_1 \otimes \underline{\mathrm{pr}}_2) \circ \Delta.$

- $\mathcal{V}^B(\mathfrak{M}_{0,5}) \curvearrowright \ker\{\mathcal{V}^B(\mathfrak{M}_{0,5}) \overset{\mathbf{pr}_5}{\twoheadrightarrow} \mathcal{V}^B\} \simeq \mathcal{V}^B(\mathfrak{M}_{0,5})^{\oplus 3}$
  $\Rightarrow \underline{\varpi} : \mathcal{V}^B(\mathfrak{M}_{0,5}) \to M_3(\mathcal{V}^B(\mathfrak{M}_{0,5}))$
  $\underline{\varpi}$ is an algebra morphism.
- $\underline{\mathbf{row}_1} := \Big((X_1 - 1) \otimes 1, \quad 1 \otimes (1 - X_1), \quad 0\Big) \in M_{1\times 3}((\mathcal{V}^B)^{\otimes 2})$
- $\underline{\mathbf{col}_1} := \begin{pmatrix} 1 \otimes 1 \\ -1 \otimes 1 \\ 0 \end{pmatrix} \in M_{3\times 1}((\mathcal{V}^B)^{\otimes 2})$

Define an algebra morphism

$$\underline{\rho} : \mathcal{V}^B \to M_3((\mathcal{V}^B)^{\otimes 2})$$

as $\mathcal{V}^B \overset{\ell}{\to} \mathcal{V}^B(\mathfrak{M}_{0,5}) \overset{\underline{\varpi}}{\to} M_3(\mathcal{V}^B(\mathfrak{M}_{0,5})) \xrightarrow{M_3(\mathbf{pr}_{12})} M_3((\mathcal{V}^B)^{\otimes 2})$.

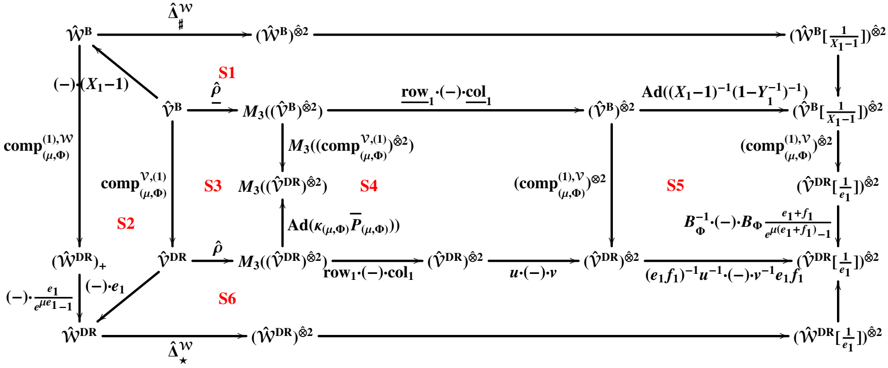
Proposition: The following diagram commutes

$$\begin{CD}
\mathcal{V}^{\mathbf{B}} @>{\underline{\rho}}>> M_3((\mathcal{V}^{\mathbf{B}})^{\otimes 2}) @>{\underline{\mathbf{row}}_1 \cdot(-)\cdot \underline{\mathbf{col}}_1}>> (\mathcal{V}^{\mathbf{B}})^{\otimes 2} \\
@V{(-)\cdot(X_1-1)}V{\simeq}V @. @VV{\mathbf{Ad}((X_1-1)^{-1}\otimes(1-X_1^{-1})^{-1})}V \\
@. @. \mathcal{V}^{\mathbf{B}}[\frac{1}{X_1-1}]^{\otimes 2} \\
@. @. @AAA \\
\mathcal{W}^{\mathbf{B},+} @>>{\Delta_\sharp^{\mathcal{W}}}> @. (\mathcal{W}^{\mathbf{B}})^{\otimes 2}
\end{CD}$$

Proof. Similar to De Rham case. □

Proof of "algebra" comparison result

Follows from the commutativity of the following diagram



 Proof of "algebra" comparison result

where

$$\mathbf{e}_1 := e_1 \otimes 1, \quad \mathbf{f}_1 := 1 \otimes e_1, \quad \mathbf{X}_1 := X_1 \otimes 1, \quad \mathbf{Y}_1 := 1 \otimes X_1,$$

$$\mathbf{u} := B_\Phi \cdot \frac{e^{\mu \mathbf{e}_1} - 1}{\mathbf{e}_1} \cdot \frac{1 - e^{-\mu \mathbf{f}_1}}{\mathbf{f}_1}, \quad \mathbf{v} = \mathbf{u}^{-1} \cdot \frac{e^{\mu(\mathbf{e}_1 + \mathbf{f}_1)} - 1}{\mathbf{e}_1 + \mathbf{f}_1},$$

$$\kappa_{(\mu,\Phi)} := e^{-(\mu/2)\mathbf{f}_1} \Phi(\mathbf{e}_0, \mathbf{e}_1) \Phi(\mathbf{f}_0, \mathbf{f}_1) \in ((\hat{\mathcal{V}}^{\mathrm{DR}})^{\hat{\otimes} 2})^\times,$$

$$P_{(\mu,\Phi)} \in \mathrm{GL}_3((U\mathfrak{p}_5)^\wedge)$$

is defined by

$$\mathrm{comp}^{((\bullet\bullet)\bullet)\bullet} \begin{pmatrix} x_{15} - 1 \\ x_{25} - 1 \\ x_{35} - 1 \end{pmatrix} = P_{(\mu,\Phi)} \cdot \begin{pmatrix} e_{15} \\ e_{25} \\ e_{35} \end{pmatrix}$$

and

$$\overline{P}_{(\mu,\Phi)} := \mathrm{pr}_{12}(P_{(\mu,\Phi)}) \in \mathrm{GL}_3((\hat{\mathcal{V}}^{\mathrm{DR}})^{\hat{\otimes} 2}).$$

 Proof of ”algebra” comparison result

Commutativity of big diagram implies that of

$$\begin{array}{ccc}
(\hat{\mathcal{W}}^{\mathrm{B}})_+ & \xrightarrow{\hat{\Delta}^{\mathcal{W}}_{\sharp}} & \hat{\mathcal{W}}^{\mathrm{B}}[\frac{1}{X_1-1}]^{\hat{\otimes}2} \\
\Big\downarrow \mathrm{comp}^{(1),\mathcal{W}}_{(\mu,\Phi)} & & \Big\downarrow (\mathrm{comp}^{(1),\mathcal{W}}_{(\mu,\Phi)})^{\otimes 2} \\
(\hat{\mathcal{W}}^{\mathrm{DR}})_+ & & \hat{\mathcal{W}}^{\mathrm{DR}}[\frac{1}{e_1}]^{\hat{\otimes}2} \\
\Big\downarrow (-)\cdot\frac{e_1}{e^{\mu e_1}-1} & & \Big\downarrow B_{\Phi}^{-1}\cdot(-)\cdot B_{\Phi}\frac{e_1+f_1}{e^{\mu(e_1+f_1)}-1} \\
(\hat{\mathcal{W}}^{\mathrm{DR}})_+ & \xrightarrow[\hat{\Delta}^{\mathcal{W}}_{\star}]{} & \hat{\mathcal{W}}^{\mathrm{DR}}[\frac{1}{e_1}]^{\hat{\otimes}2}
\end{array}$$

which by $\Delta^{\mathcal{W}}_{\star}(\frac{e_1}{e^{\mu e_1}-1}) = \frac{e_1+f_1}{e^{\mu(e_1+f_1)}-1}$ implies comm. of ”algebra” comparison diagram.

Proof of "algebra" comparison result

- Comm. of S1, S6: geometric interpretations of $\boldsymbol{\Delta}_\sharp^{\mathcal{W}}$, $\boldsymbol{\Delta}_\star^{\mathcal{W}}$.
- Comm. of S2: algebra morphism nature of $\mathbf{comp}_{(\mu,\Phi)}^{(1),\mathcal{V}}$, its compatibility with $\mathbf{comp}_{(\mu,\Phi)}^{(1),\mathcal{W}}$, its property $X_1 \mapsto e^{\mu e_1}$.
- Comm. of S5: same properties of $\mathbf{comp}_{(\mu,\Phi)}^{(1),\mathcal{V}}$, identities relating u, B_Φ, e_1, f_1 and v, B_Φ, e_1, f_1.
- Comm. of S3: ρ (resp. $\underline{\rho}$) is based on choice of basis $(e_{i5})_{i=1,2,3}$ (resp. $(x_{i5} - 1)_{i=1,2,3}$) for $\mathbf{ker}(U\mathfrak{p}_5 \to \mathcal{V}^{\mathrm{DR}})$ (resp. $\mathbf{ker}(\mathbb{C}P_5 \to \mathcal{V}^{\mathrm{B}})$), and $P_{(\mu,\Phi)}$ expresses comparison of these bases.

 Proof of "algebra" comparison result

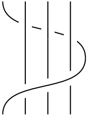
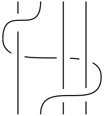
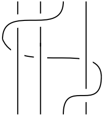
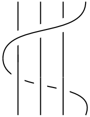
- Comm. of S4. is a consequence of the equalities

$$(\mathrm{comp}^{(1),\nu}_{(\mu,\Phi)})^{\otimes 2}(\underline{\mathrm{col}}_1) = \kappa_{(\mu,\Phi)}\overline{P}_{(\mu,\Phi)} \cdot \mathrm{col}_1 \cdot \nu,$$

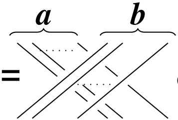
$$(\mathrm{comp}^{(1),\nu}_{(\mu,\Phi)})^{\otimes 2}(\underline{\mathrm{row}}_1) = u \cdot \mathrm{row}_1 \cdot (\kappa_{(\mu,\Phi)}\overline{P}_{(\mu,\Phi)})^{-1},$$

whose proofs necessitate *explicit computation:*

(a) one expresses the braid group elements x_{i5}

$x_{15} =$ , $x_{25} =$ , $x_{35} =$ , $x_{45} =$ 

as products of $\sigma_{a,b}$

$\sigma_{a,b} =$ 

$\in B_{a+b}$

(b) this enables one to compute explicitly $\mathbf{comp}^{((\bullet\bullet)\bullet)\bullet}_{(\mu,\Phi)}(x_{i5}-1)$ *as elements of* $U(\mathfrak{f}_3)^\wedge = \mathbb{C}\langle\langle e_{i5}, i=1,\ldots,4\rangle\rangle$;
(c) one derives from there the computation of $P_{(\mu,\Phi)}$;
(d) one further derives the computation of $\overline{P}_{(\mu,\Phi)}$;
(e) one plugs the obtained value into the first identity;
(f) the second identity can be similarly obtained. □

Section 4: Proof of ”module” comparison result

Interpretation of $\Delta_\star^{\mathcal{M}}$ in terms of moduli spaces $\mathfrak{M}_{0,4}$ and $\mathfrak{M}_{0,5}$

Set

$$\mathbf{col}_0 := \begin{pmatrix} 0 \\ -e_1 \cdot 1_{\mathrm{DR}}^{\otimes 2} \\ e_1 \cdot 1_{\mathrm{DR}}^{\otimes 2} \end{pmatrix} \in M_{3\times 1}((\mathcal{M}^{\mathrm{DR}})^{\otimes 2}).$$

Proposition: *The following diagram commutes*

$$\begin{array}{ccccc} \mathcal{V}^{\mathrm{DR}} & & \xrightarrow{\rho} & & M_3((\mathcal{V}^{\mathrm{DR}})^{\otimes 2}) \\ \downarrow \mathbf{can} & & & & \downarrow (e_1 f_1)^{-1}\mathbf{row}_1 \cdot (-) \cdot \mathbf{col}_0 \\ \mathcal{M}^{\mathrm{DR}} & \xrightarrow[\Delta_\star^{\mathcal{M}}]{} & (\mathcal{M}^{\mathrm{DR}})^{\otimes 2} & \longrightarrow & (\mathcal{M}^{\mathrm{DR}}[\frac{1}{e_1}])^{\otimes 2} \end{array}$$

Proof.

(a) Show that $\rho(e_0)\cdot \mathbf{col}_0 = \mathbf{0}$.

(b) Derive the existence of map $\delta : \mathcal{M}^{\mathrm{DR}} \to (\mathcal{M}^{\mathrm{DR}}[\frac{1}{e_1}])^{\otimes 2}$ such that

$$\begin{array}{ccc} \mathcal{V}^{\mathrm{DR}} & \xrightarrow{\ \rho\ } & M_3((\mathcal{V}^{\mathrm{DR}})^{\otimes 2}) \\ {\scriptstyle \mathbf{can}}\downarrow & & \downarrow {\scriptstyle (e_1 f_1)^{-1}\mathbf{row}_1\cdot(-)\cdot\mathbf{col}_0} \\ \mathcal{M}^{\mathrm{DR}} & \xrightarrow[\ \delta\ ]{} & (\mathcal{M}^{\mathrm{DR}}[\frac{1}{e_1}])^{\otimes 2} \end{array}$$

commutes.

(c) Using $\rho(e_1) = \mathbf{col}_1\cdot \mathbf{row}_1$, show that δ is compatible with $\Delta_\star^{\mathcal{W}} : \mathcal{W}^{\mathrm{DR}} \to (\mathcal{W}^{\mathrm{DR}})^{\otimes 2}$ and module structure of $\mathcal{M}^{\mathrm{DR}}$ over $\mathcal{W}^{\mathrm{DR}}$.

(d) Compute $(e_1 f_1)^{-1}\mathbf{row}_1\cdot \mathbf{col}_0 = \mathbf{1}_{\mathrm{DR}}^{\otimes 2}$ to get $\delta(\mathbf{1}_{\mathrm{DR}}) = \mathbf{1}_{\mathrm{DR}}^{\otimes 2}$.

(e) Derive $\delta = \Delta_\star^{\mathcal{M}}$. □

Interpretation of $\Delta_\sharp^{\mathcal{M}}$ in terms of moduli spaces $\mathfrak{M}_{0,4}$ and $\mathfrak{M}_{0,5}$

Set

$$\underline{\mathbf{col}}_0 := \begin{pmatrix} \mathbf{0} \\ (1 - X_1)Y_1^{-1} \cdot 1_{\mathrm{B}}^{\otimes 2} \\ (1 - X_1^{-1})Y_1^{-1} \cdot 1_{\mathrm{B}}^{\otimes 2} \end{pmatrix} \in M_{3\times 1}((\mathcal{M}^{\mathrm{B}})^{\otimes 2}).$$

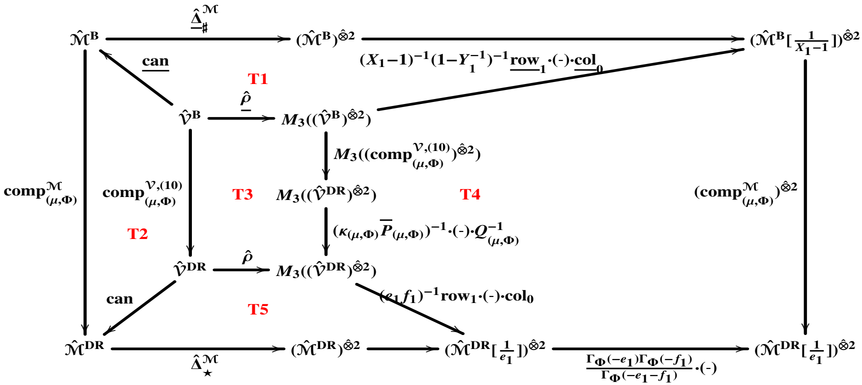
Proposition: The following diagram commutes

$$\begin{array}{ccccc}
\mathcal{V}^{\mathrm{B}} & & \xrightarrow{\quad \underline{\rho} \quad} & & M_3((\mathcal{V}^{\mathrm{B}})^{\otimes 2}) \\
\downarrow{\scriptstyle \underline{\mathrm{can}}} & & & & \downarrow{\scriptstyle (X_1-1)^{-1}(1-Y_1^{-1})^{-1}\underline{\mathbf{row}}_1\cdot(-)\cdot\underline{\mathbf{col}}_0} \\
\mathcal{M}^{\mathrm{B}} & \xrightarrow[\underline{\Delta}_\sharp^{\mathcal{M}}]{} & (\mathcal{M}^{\mathrm{B}})^{\otimes 2} & \longrightarrow & (\mathcal{M}^{\mathrm{B}}[\frac{1}{X_1-1}])^{\otimes 2}
\end{array}$$

Proof. Similar to De Rham case. □

Proof of "module" comparison result

Follows from the commutativity of the following diagram



where $Q_{(\mu,\Phi)} \in \mathrm{GL}_3((\hat{\mathcal{V}}^{\mathrm{DR}})^{\hat{\otimes}2})$ is given by

$$Q^{-1}_{(\mu,\Phi)} = \Phi(e_1,e_0)\Phi(f_1,f_0)\cdot\kappa_{(\mu,\Phi)}\overline{P}_{\mu,\Phi}\cdot\hat{\rho}(\Phi).$$

- Comm. of T1 and T5: by the above geometric interpretations of $\Delta_\star^{\mathcal{M}}$, $\Delta_\sharp^{\mathcal{M}}$.
- Comm. of T2: by construction.
- Comm. of T3: states equality of two maps $\hat{\mathcal{V}}^{\mathbf{B}} \to M_3((\hat{\mathcal{V}}^{\mathbf{DR}})^{\hat{\otimes}2})$.
  These maps are module morphisms over two algebra morphisms $\hat{\mathcal{V}}^{\mathbf{B}} \to M_3((\hat{\mathcal{V}}^{\mathbf{DR}})^{\hat{\otimes}2})$ which are two parts of S3 and turn out to be equal due to comm. of S3.
  The value taken by $Q_{(\mu,\Phi)}$ guarantees that these maps agree on generator $\mathbf{1} \in \mathcal{V}^{\mathbf{DR}}$.

- Comm. of T4 is a consequence of the equalities

$$(\mathbf{comp}^{(1),\mathcal{V}}_{(\mu,\Phi)})^{\otimes 2}((X_1-1)^{-1}(1-Y_1^{-1})^{-1}\underline{\mathbf{row}}_1) = \frac{\Gamma_\Phi(-e_1)\Gamma_\Phi(-f_1)}{\Gamma_\Phi(-e_1-f_1)}(e_1f_1)^{-1}\cdot \mathbf{row}_1\cdot(\kappa_{(\mu,\Phi)}\overline{P}_{(\mu,\Phi)})^{-1}.$$

and

$$(\mathbf{comp}^{(10),\mathcal{M}}_{(\mu,\Phi)})^{\otimes 2}(\underline{\mathbf{col}}_0) = (\mathbf{comp}^{(10),\mathcal{V}}_{(\mu,\Phi)})^{\otimes 2}(1)\cdot\kappa_{(\mu,\Phi)}Q^{-1}_{(\mu,\Phi)}\cdot\mathbf{col}_0.$$

First equality: an immediate consequence of already proved equality (proof of S4).
Second equality: *explicit computation* parallel to previous one but dealing with (•(••))• rather than ((••)•)•:

(a) one expresses the braid group elements x_{i5} as products of $\sigma_{a,b}$ (same expressions as before except for x_{25});
(b) this enables one to compute explicitly $\mathbf{comp}^{(\bullet(\bullet\bullet))\bullet}_{(\mu,\Phi)}(x_{i5}-1)$ *as elements of* $U(\mathfrak{f}_3)^\wedge = \mathbb{C}\langle\langle e_{i5}, i=1,\ldots,4\rangle\rangle$;
(c) one derives the computation of $R_{(\mu,\Phi)} \in \mathbf{GL}_3((U\mathfrak{p}_5)^\wedge)$ defined by

$$\mathbf{comp}^{(\bullet(\bullet\bullet))\bullet}\begin{pmatrix} x_{15}-1 \\ x_{25}-1 \\ x_{35}-1 \end{pmatrix} = R_{(\mu,\Phi)} \cdot \begin{pmatrix} e_{15} \\ e_{25} \\ e_{35} \end{pmatrix}$$

(d) and therefore of $\overline{R}_{(\mu,\Phi)} := \mathbf{pr}_{12}(R_{(\mu,\Phi)}) \in \mathbf{GL}_3((\hat{\mathcal{V}}^{\mathrm{DR}})^{\hat{\otimes}2})$.
(e) one proves that $Q^{-1}_{(\mu,\Phi)} = \Phi(e_1,e_0)\Phi(f_1,f_0)\kappa_{(\mu,\Phi)}\overline{R}_{(\mu,\Phi)}$
(f) one then computes $Q^{-1}_{(\mu,\Phi)}$.
(g) one uses the obtained expression of $Q^{-1}_{(\mu,\Phi)}$ to explicitly prove wanted equality. □