The structure of categorified representation theory

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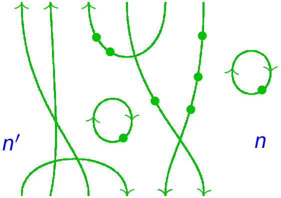
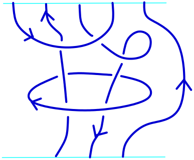


$$\mathcal{E}^3\mathcal{F}^3 1_n\{s\} \Longrightarrow \mathcal{E}^3\mathcal{F}^3 1_n\{s'\}$$

May 23rd, 2011

Quantum link invariants

Jones polynomial



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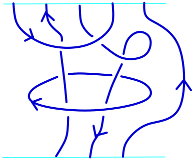

Representation theory of quantum $\mathfrak{sl}_2$

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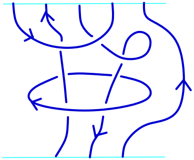
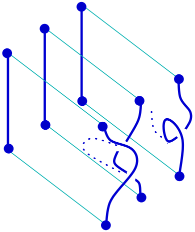


Representation theory of quantum $\mathfrak{sl}_2$

Categorification



Khovanov homology

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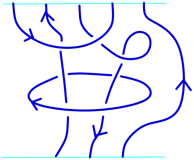

Representation theory of quantum $\mathfrak{sl}_2$

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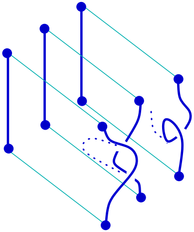

Categorified representation theory

Khovanov homology

Quantum sl(2)

The quantum group $\mathbf{U}_q(\mathfrak{sl}_2)$ is the associative algebra (with unit) over $\mathbb{C}(q)$ with generators E, F, K, K^{-1} and relations

- $KK^{-1} = 1 = K^{-1}K,$
- $KE = q^2EK, \qquad KF = q^{-2}FK,$
- $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$

A finite-dimensional representation V of $\mathbf{U}_q(\mathfrak{sl}_2)$ is given by homomorphism

$$\mathbf{U}_q(\mathfrak{sl}_2) \longrightarrow \mathrm{End}(V)$$

for some $\mathbb{C}(q)$-vector space V.

We can decompose V into eigenspaces for the action of K.

$$V = \bigoplus_{n\in\mathbb{Z}} V(n)$$

$$Kv = q^n v \qquad v \in V(n)$$

Representation theory of $\mathbf{U}_q(\mathfrak{sl}_2)$

A representation of $\mathbf{U}_q(\mathfrak{sl}_2)$ is a collection of

$$V_{-N} \quad \cdots \quad V_{n-2} \quad V_n \quad V_{n+2} \quad \cdots \quad V_N$$

- vector spaces ($Kv = q^n v$ for $v \in V_n$),

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$$\begin{tikzcd} V_{-N} \arrow[r, bend left, "E"] & \cdots \arrow[l, bend left, "F"] & V_{n-2} \arrow[r, bend left, "E"] & V_n \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & V_{n+2} \arrow[l, bend left, "F"] & \cdots \arrow[r, bend left, "E"] & V_N \arrow[l, bend left, "F"] \end{tikzcd}$$

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- satisfying $EFv - FEv = [n]v$ for $v \in V_n$.

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$$

Categorical representations of $\mathbf{U}_q(\mathfrak{sl}_2)$

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- satisfying

$$\mathcal{E}\mathcal{F}\mathbf{1}_n \cong \mathcal{F}\mathcal{E}\mathbf{1}_n \bigoplus_{[n]} \mathbf{1}_n \qquad \text{for } n \geq 0$$

$$\mathcal{F}\mathcal{E}\mathbf{1}_n \cong \mathcal{E}\mathcal{F}\mathbf{1}_n \bigoplus_{[-n]} \mathbf{1}_n \qquad \text{for } n \leq 0$$

Example (cohomology of partial flag varieties)

Fix $N > 0$ consider the varieties

$$
\begin{tikzcd}
\underset{\{0\subset\mathbb{C}^k\subset\mathbb{C}^N\}}{Gr(k,N)} & \underset{\{0\subset\mathbb{C}^k\subset\mathbb{C}^{k+1}\subset\mathbb{C}^N\}}{Fl(k,k+1,N)} \arrow[l] \arrow[r] & \underset{\{0\subset\mathbb{C}^{k+1}\subset\mathbb{C}^N\}}{Gr(k+1,N)}
\end{tikzcd}
$$

which give rise to inclusions

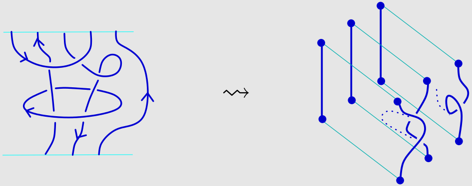
$$
\begin{tikzcd}
H_k := H^*(Gr(k,N)) \arrow[r] & H_{k,k+1} := H^*(Fl(k,k+1,N)) & H_{k+1} := H^*(Gr(k+1,N)) \arrow[l]
\end{tikzcd}
$$

on cohomology making $H_{k,k+1}$ an (H_k, H_{k+1})-bimodule.

$$
\begin{tikzcd}
H_0 \arrow[r, bend left, "H_{1,0}"] & \arrow[l, bend left, "H_{0,1}"] \quad \cdots H_{k-1} \arrow[r, bend left, "H_{k,k-1}"] & H_k \arrow[l, bend left, "H_{k-1,k}"] \arrow[r, bend left, "H_{k+1,k}"] & H_{k+1} \cdots \arrow[l, bend left, "H_{k,k+1}"] \quad \arrow[r, bend left, "H_{N,N-1}"] & H_k \arrow[l, bend left, "H_{N-1,N}"]
\end{tikzcd}
$$

Higher structure

We expect to see a new level of structure in categorification.



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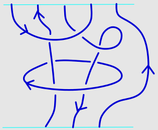

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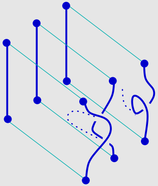

What is the higher structure of categorical representation theory?

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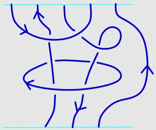

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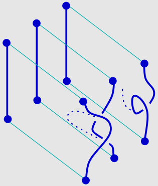

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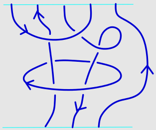

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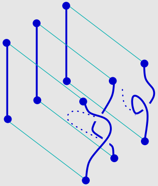

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Idea: look at the structure of natural transformations between functors in categorical $\mathbf{U}_q(\mathfrak{sl}_2)$-actions.

The $\mathfrak{sl}_2$-relations should follow as consequences of this higher structure.

This suggests a categorification of the $\mathbf{U}_q(\mathfrak{sl}_2)$ that allows for maps between algebra elements.

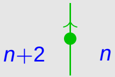
$\mathcal{U}$ is an additive $\Bbbk$-linear 2-category with

- objects: n for $n \in \mathbb{Z}$.
- 1-morphism in $\mathcal{U}$ from n to n' is a formal finite direct sum of

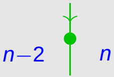
$$\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\cdots\mathcal{E}^{\alpha_m}\mathcal{F}^{\beta_m}1_n\{s\} = 1_{n'}\mathcal{E}^{\alpha_1}\mathcal{F}^{\beta_1}\cdots\mathcal{E}^{\alpha_m}\mathcal{F}^{\beta_m}1_n\{s\}$$

  for any $s \in \mathbb{Z}$ and $n' = n + 2\sum \alpha_i - 2\sum \beta_i$.
- 2-morphisms given by $\Bbbk$-linear combinations of diagrams

Generating 2-morphisms



$: \mathcal{E}\mathbf{1}_n\{2\} \to \mathcal{E}\mathbf{1}_n$



$: \mathcal{F}\mathbf{1}_n\{2\} \to \mathcal{F}\mathbf{1}_n$

$: \mathcal{E}\mathcal{E}\mathbf{1}_n\{-2\} \to \mathcal{E}\mathcal{E}\mathbf{1}_n$

$: \mathcal{F}\mathcal{F}\mathbf{1}_n\{-2\} \to \mathcal{F}\mathcal{F}\mathbf{1}_n$

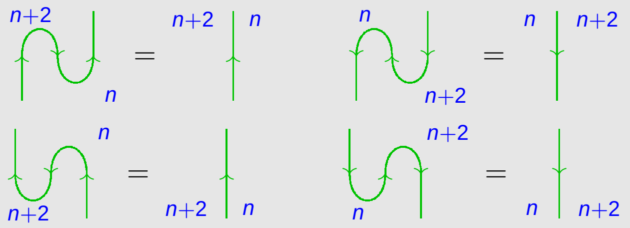
$: \mathbf{1}_n\{1+n\} \to \mathcal{F}\mathcal{E}\mathbf{1}_n$

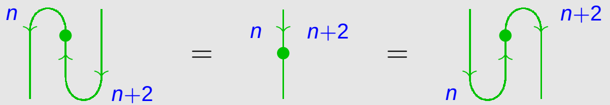
$: \mathbf{1}_n\{1-n\} \to \mathcal{E}\mathcal{F}\mathbf{1}_n$

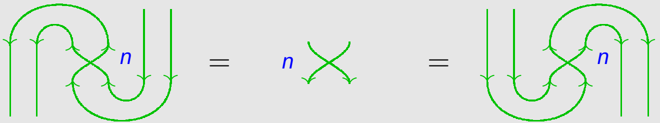
$: \mathcal{F}\mathcal{E}\mathbf{1}_n\{1+n\} \to \mathbf{1}_n$

$: \mathcal{E}\mathcal{F}\mathbf{1}_n\{1-n\} \to \mathbf{1}_n$

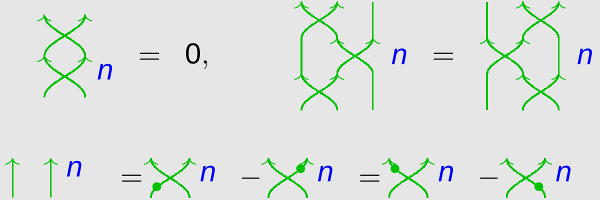
Topological invariance







NilHecke relations



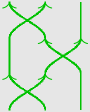
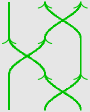
Positivity of bubbles

All dotted bubbles of negative degree are zero. That is,

deg () $= 2(1-n)+2\beta$ deg () $= 2(1+n)+2\beta$

NilHecke relations

$$
\text{} \; {}_{n} \;=\; 0, \qquad
\text{} \; n \;=\; \text{} \; n
$$

 n $=$ $0,$  n $=$  n

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deg $\Bigg($  $\Bigg) = 2(1-n) + 2\beta$ deg $\Bigg($  $\Bigg) = 2(1+n) + 2\beta$

$\Rightarrow$  $= \; 0 \quad \text{if } \beta < n-1$  $= \; 0 \quad \text{if } \beta < -n-1$

$\mathfrak{sl}_2$ isomorphism $\mathcal{F}\mathcal{E}1_n \oplus_{[n]} 1_n \cong \mathcal{E}\mathcal{F}1_n$ for $n \geq 0$

Recall that for $[n] = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$ we write
$\oplus_{[n]}1_n := 1_n\{n-1\} \oplus 1_n\{n-3\} \oplus \cdots \oplus 1_n\{1-n\}$.

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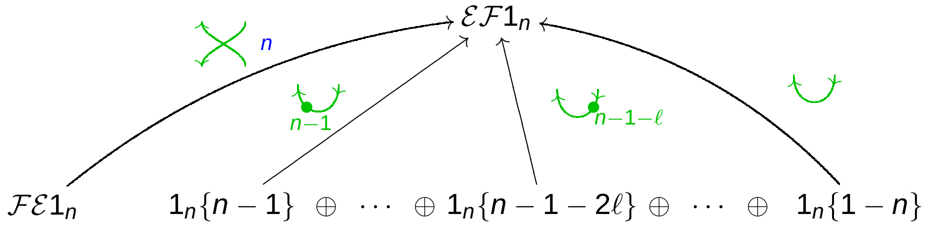
We require that the map



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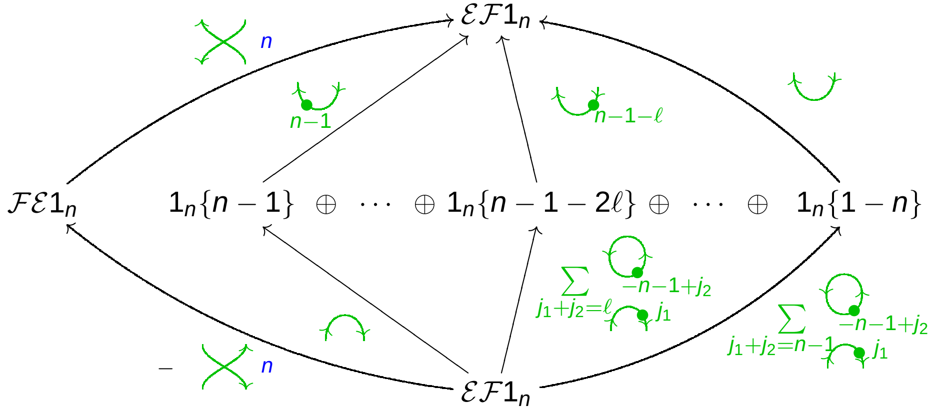


has a specified inverse. Similarly for $n \leq 0$.

Relations in $\mathcal{U}$

The previous relations imposed on $\mathcal{U}$ were

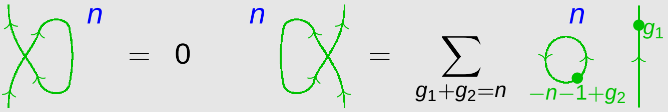
- $\mathcal{E}\mathbf{1}_n$ and $\mathcal{F}\mathbf{1}_n$ are biadjoint up to grading shift.
- All 2-morphisms are cyclic with respect to this biadjoint structure (topological invariance, or planar algebra condition)
- The nilHecke algebra acts on $\mathrm{END}(\mathcal{E}^a)\mathbf{1}_n$.
- Negative degree bubbles are zero.

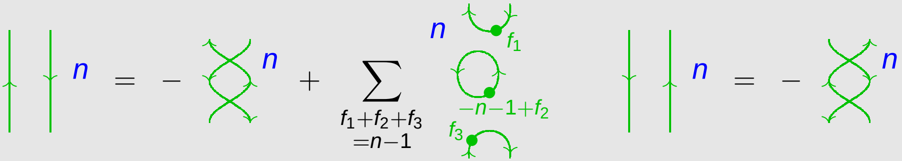
The requirement that the specified maps give isomorphisms

$$\begin{aligned} \mathcal{E}\mathcal{F}1_n &\cong \mathcal{F}\mathcal{E}1_n \oplus_{[n]} 1_n \qquad && \text{for } n \geq 0 \\ \mathcal{F}\mathcal{E}1_n &\cong \mathcal{E}\mathcal{F}1_n \oplus_{[-n]} 1_n \qquad && \text{for } n \leq 0 \end{aligned}$$

for the $\mathfrak{sl}_2$ relations imposes diagrammatic relations on the 2-category $\mathcal{U}$ that depend on the weight space n.

For $n > 0$





Similar relations for $n < 0$.

For $n = 0$

Theorem (L.)

This graphical calculus is consistent and categorifies the integral idempotent version $\dot{\mathbf{U}}_{\mathbb{Z}}$ of $\mathbf{U}_q(\mathfrak{sl}_2)$. This $\mathbb{Z}[q,q^{-1}]$-algebra is obtained from $\mathbf{U}_q(\mathfrak{sl}_2)$ by replacing 1 by mutually orthogonal idempotents 1_n projecting onto the nth weight space.

- $\dot{\mathbf{U}}_{\mathbb{Z}} \cong K_0(\dot{\mathcal{U}})$ the Grothendieck ring/category of this 2-category

$$x \oplus y \in \dot{\mathcal{U}} \;\rightsquigarrow\; [x]+[y] \in K_0(\dot{\mathcal{U}}) \qquad x\{s\} \;\rightsquigarrow\; q^s[x] \in K_0(\dot{\mathcal{U}})$$

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- Indecomposable 1-morphisms $\Leftrightarrow$ Lusztig canonical basis element
- Graded 2Hom $\mathrm{HOM}_{\dot{\mathcal{U}}}(x,y)$ categorifies the semilinear form $\langle x,y\rangle$
- The 2-category $\dot{\mathcal{U}}$ acts on cohomology of iterated flag varieties, categorifying the irreducible N-dimensional rep of $\mathbf{U}_q(\mathfrak{sl}_2)$

Theorem (L.)

This graphical calculus is consistent and categorifies the integral idempotent version $\dot{\mathbf{U}}_{\mathbb{Z}}$ of $\mathbf{U}_q(\mathfrak{sl}_2)$. This $\mathbb{Z}[q,q^{-1}]$-algebra is obtained from $\mathbf{U}_q(\mathfrak{sl}_2)$ by replacing 1 by mutually orthogonal idempotents 1_n projecting onto the nth weight space.

- $\dot{\mathbf{U}}_{\mathbb{Z}} \cong K_0(\dot{\mathcal{U}})$ the Grothendieck ring/category of this 2-category

$$x \oplus y \in \dot{\mathcal{U}} \;\rightsquigarrow\; [x]+[y] \in K_0(\dot{\mathcal{U}}) \qquad x\{s\} \;\rightsquigarrow\; q^s[x] \in K_0(\dot{\mathcal{U}})$$

- Indecomposable 1-morphisms $\Leftrightarrow$ Lusztig canonical basis element
- Graded 2Hom $\mathrm{HOM}_{\dot{\mathcal{U}}}(x,y)$ categorifies the semilinear form $\langle x,y\rangle$
- The 2-category $\dot{\mathcal{U}}$ acts on cohomology of iterated flag varieties, categorifying the irreducible N-dimensional rep of $\mathbf{U}_q(\mathfrak{sl}_2)$

Theorem (Khovanov, L., Mackaay, Stošić)

The natural homomorphism $\dot{\mathbf{U}}_{\mathbb{Z}} \to K_0(\dot{\mathcal{U}})$ remains an isomorphism when considering $\Bbbk = \mathbb{Z}$-linear combinations of 2-morphisms.

2-representations

Definition

Let $\mathcal{K}$ be a graded additive $\Bbbk$-linear 2-category. A 2-representation is a 2-functor $\mathcal{U} \to \mathcal{K}$.

Example ($\mathcal{K} = \mathit{Cat}$)

$$
\begin{aligned}
\mathcal{U} &\longrightarrow \mathit{Cat} \\
n &\mapsto \text{category } \mathcal{V}_n \\
\mathcal{E}\mathbf{1}_n &\mapsto \text{functors } \mathcal{E}\mathbf{1}_n \colon \mathcal{V}_n \to \mathcal{V}_{n+2} \\
\mathcal{F}\mathbf{1}_n &\mapsto \text{functors } \mathcal{F}\mathbf{1}_n \colon \mathcal{V}_n \to \mathcal{V}_{n-2} \\
\text{generating 2-morphisms} &\mapsto \text{natural transformations} \\
\text{relations} &\mapsto \text{relations between natural transformations}
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Do we really have to check all those relations?

Theorem (Cautis-L.)

Given a finite dimensional categorical representation of $\mathbf{U}_q(\mathfrak{sl}_2)$ (functors $\mathcal{E}$ and $\mathcal{F}$ satisfying $\mathfrak{sl}_2$-relations up to isomorphism) then this structure extends to a 2-representation of $\mathcal{U}$ if the following additional conditions are satisfied:

- The functors $\mathcal{E}$ and $\mathcal{F}$ are biadjoint up to grading shift.
- The nilHecke algebra axioms hold.
- The graded vector spaces $\mathrm{HOM}(\mathbf{1}_n, \mathbf{1}_n)$ are zero dimensional in negative degrees and one-dimensional in degree zero.

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If much of the structure of categorified representation theory comes for free, what are the advantages of this higher structure?

Higher relations play a key role in categorification of other representation theoretic constructions.

Derived equivalences

- Chuang and Rouquier showed that derived equivalences could be constructed in the context of abelian categories using a related approach to higher representation theory.
- Cautis, Kamnitzer and Licata showed that the higher structure of $\mathcal{U}$ gives derived equivalences in the more general setting of triangulated categories.

$$\begin{tikzcd}
\cdots & \mathcal{V}_{-n+2} \arrow[r, shift left, "\mathcal{E}"] & \mathcal{V}_{-n} \arrow[l, shift left, "\mathcal{F}"] \arrow[r, shift left, "\mathcal{E}"] \arrow[rrrr, bend left, leftrightarrow] & \mathcal{V}_{-n-2} \arrow[l, shift left, "\mathcal{F}"] & \cdots & \mathcal{V}_{n-2} \arrow[r, shift left, "\mathcal{E}"] & \mathcal{V}_{n} \arrow[l, shift left, "\mathcal{F}"] \arrow[r, shift left, "\mathcal{E}"] & \mathcal{V}_{n+2} \arrow[l, shift left, "\mathcal{F}"] & \cdots
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\end{tikzcd}$$

- Used by CR to prove the Abelian defect conjecture for symmetric groups.
- Used by CKL to construct derived equivalences between derived categories of coherent sheaves on cotangent bundles to Grassmannians.

There is a complex that categorified the integral idempotent version of the Casimir element

$$\dot{C} = \prod_{n\in\mathbb{Z}} C1_n,$$
$$C1_n = 1_nC := (-q^2+2-q^{-2})EF1_n - (q^{n-1}+q^{1-n})1_n,$$

$$\mathcal{C}\mathbf{1}_n := \begin{array}{ccccc} \mathcal{E}\mathcal{F}\mathbf{1}_n\{2\} & \longrightarrow & \underline{\mathcal{E}\mathcal{F}\mathbf{1}_n} & \xrightarrow{-} & \mathcal{E}\mathcal{F}\mathbf{1}_n\{-2\} \\ \oplus & \times & \oplus & \times & \oplus \\ \mathbf{1}_n\{1-n\} & \longrightarrow & \underline{\mathcal{E}\mathcal{F}\mathbf{1}_n} & \xrightarrow[-]{} & \mathbf{1}_n\{n-1\} \end{array}$$

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\bigoplus & \bigoplus & \bigoplus \\
\mathbf{1}_n\{1-n\} \arrow[r] \arrow[uur] & \underline{\mathcal{E}\mathcal{F}\mathbf{1}_n} \arrow[r, "-"'] \arrow[uur] & \mathbf{1}_n\{n-1\}
\end{tikzcd}$$

- The Casimir complex categorifies the Casimir element of $\dot{\mathbf{U}}$.
- The Casimir complex commute with complexes in $Com(\mathcal{U})$ up to chain homotopy.

Combinatorics of symmetric functions

The relations arising from the invertibility of the $\mathfrak{sl}_2$ relations have surprising connections to symmetric functions.

Symmetric polynomials

Elementary symmetric polynomial $e_r =$ sum over all products of r distinct variables x_i

- $e_1 = x_1 + x_2 + x_3 + \dots$
- $e_2 = x_1x_2 + x_1x_3 + x_2x_3 + \dots$

Complete symmetric polynomials $h_r =$ sum over all monomials of total degree r

- $h_1 = e_1 = x_1 + x_2 + x_3 + \dots$
- $h_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 + \dots$

These two sets of functions are related by

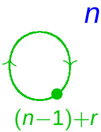
$$\sum_{r=0}^{m} (-1)^r e_r h_{m-r} = 0$$

Both provide a basis for the ring of symmetric polynomials Λ

$$\Lambda = \mathbb{Z}[e_1, e_2, e_3, \dots], \qquad \Lambda = \mathbb{Z}[h_1, h_2, h_3, \dots].$$

There is an isomorphism

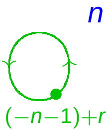
$$\phi_n : \mathrm{HOM}_{\mathcal{U}}(1_n, 1_n) \;\to\; \Lambda$$



$$\mapsto \; e_r \qquad \text{for } n \geq 0.$$

Furthermore, using fake bubbles there is also an isomorphism

$$\phi_n : \mathrm{HOM}_{\mathcal{U}}(1_n, 1_n) \;\to\; \Lambda$$



$$\mapsto \; (-1)^r h_r \qquad \text{for } n \geq 0,$$

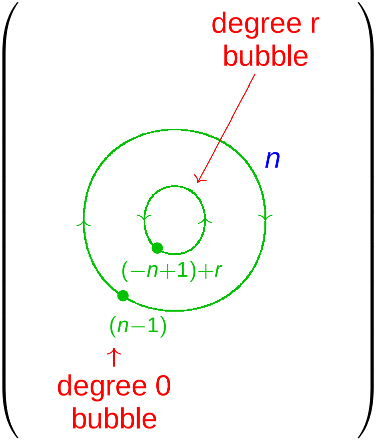
Infinite Grassmannian relation $\Rightarrow$ $\sum_{r=0}^{m} (-1)^r e_r h_{m-r} = 0$

The rth power sum is given by

$$p_r = \sum x_i^r .$$

Power sums arise naturally in the graphical calculus

ϕ_n



$= p_r$

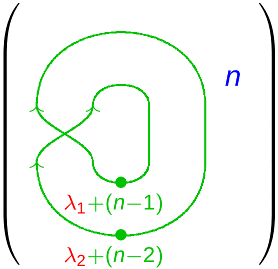
as a degree r counter-clockwise oriented bubble inside a degree 0 clockwise oriented bubble

Schur polynomials

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$, denote the conjugate partition to λ as λ'. The Schur polynomial s_λ can be expressed as

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq m}, \qquad s_\lambda = \det(\epsilon_{\lambda'_i - i + j})_{1 \leq i,j \leq \lambda_1}.$$

ϕ_n



$= \quad s_{\lambda_1, \lambda_2}$

This generalizes to arbitrary Schur polynomials.