CATEGORIFICATIONS OF THE ARROW POLYNOMIAL

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with
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Vassily Manturov
Aaron Kaestner

Title: An Extended Bracket Polynomial for Virtual Knots and Links, JKTR, Vol. 18, No. 10, Oct. 2009.
Authors: Louis H. Kauffman

arXiv:0712.2546

Title: Virtual Crossing Number and the Arrow Polynomial
Authors: H. A. Dye, Louis H. Kauffman

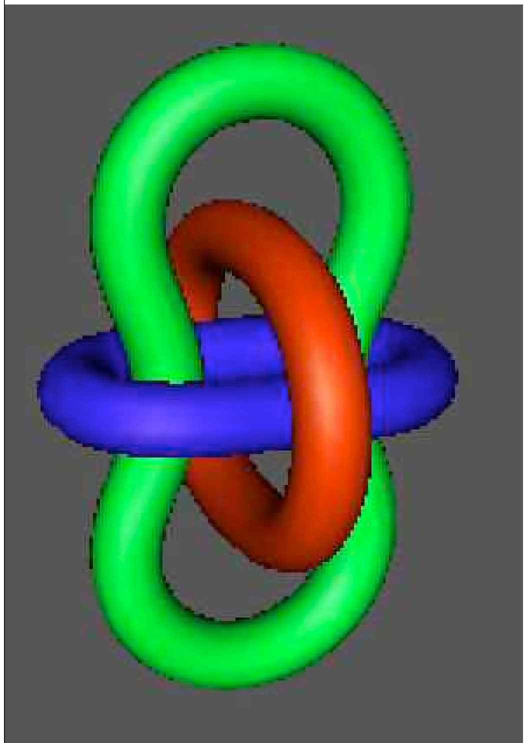
arXiv:0810.3858

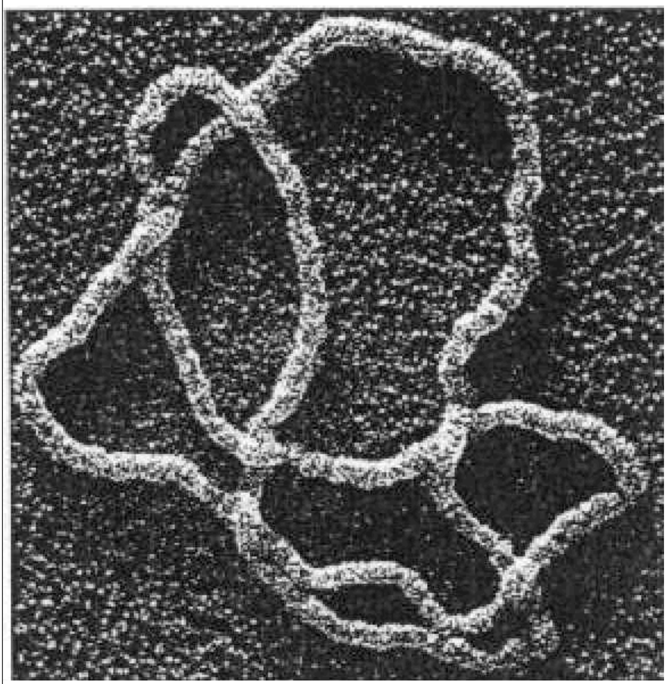
Title: On two categorifications of the arrow polynomial for virtual knots
Authors: Heather Ann Dye, Louis Hirsch Kauffman, Vassily Olegovich Manturov

arXiv:0906.3408

Title: Arrow Categorifications -- Examples and Computations
Authors: Aaron Kaestner and L.H. Kauffman (in preparation).

AND ...





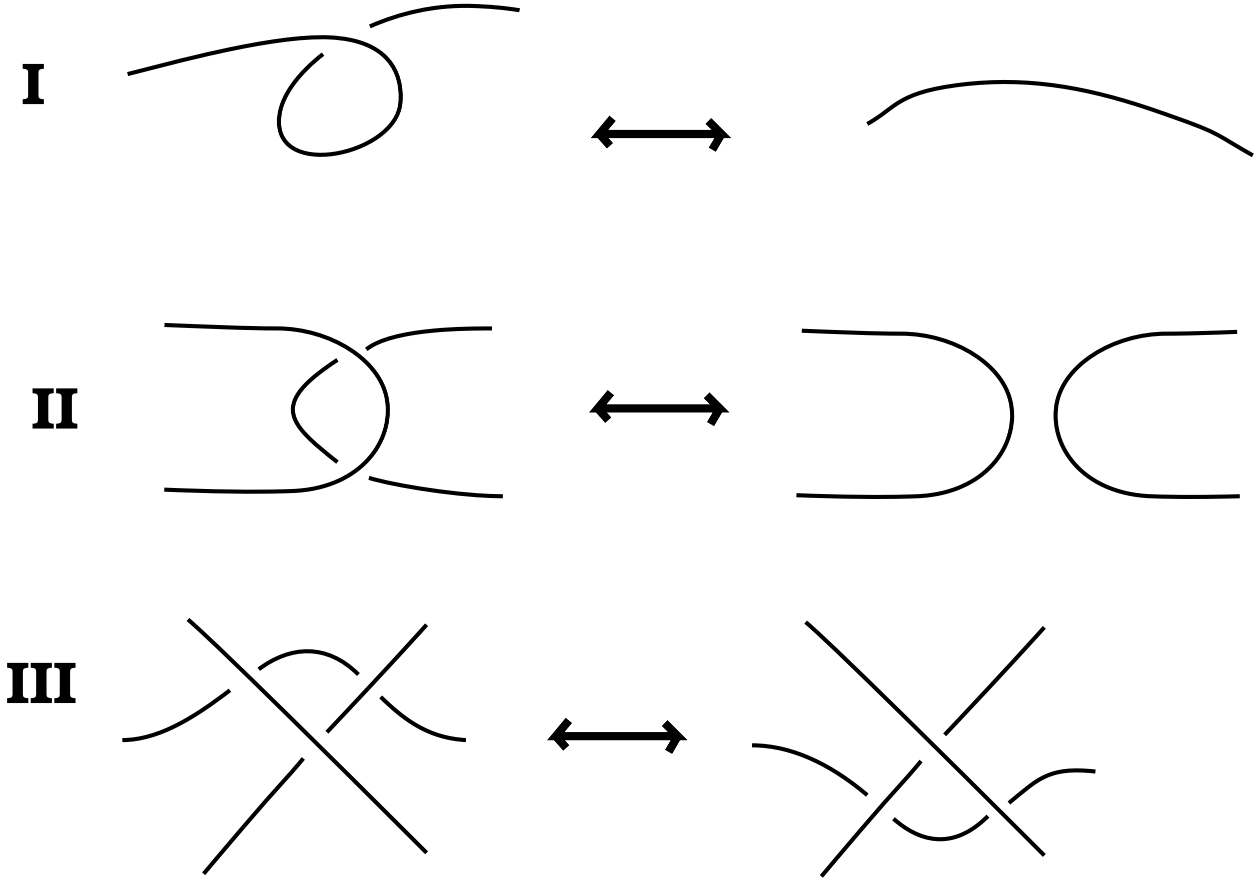
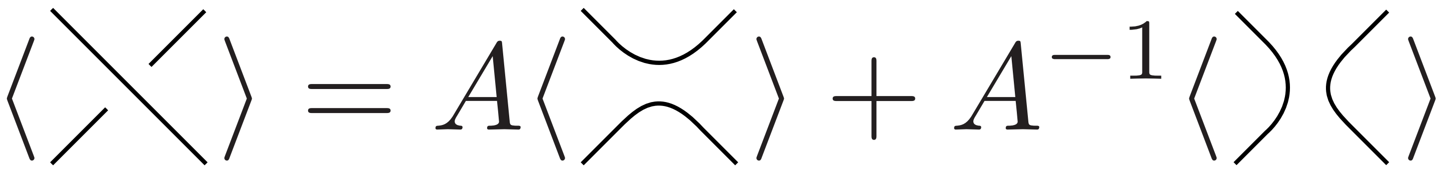


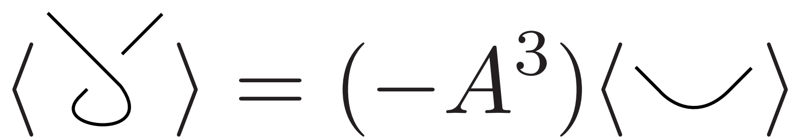
Figure 2 - The Reidemeister Moves.

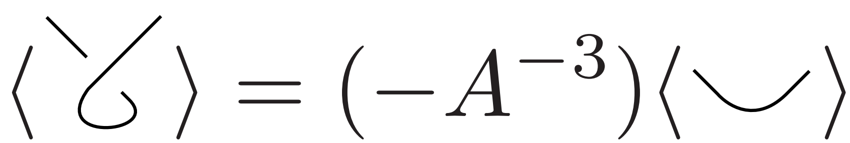
Reidemeister Moves reformulate knot theory in terms of graph combinatorics.

Bracket Polynomial Model for the Jones Polynomial



$$\langle K \bigcirc \rangle = (-A^2 - A^{-2})\langle K \rangle$$



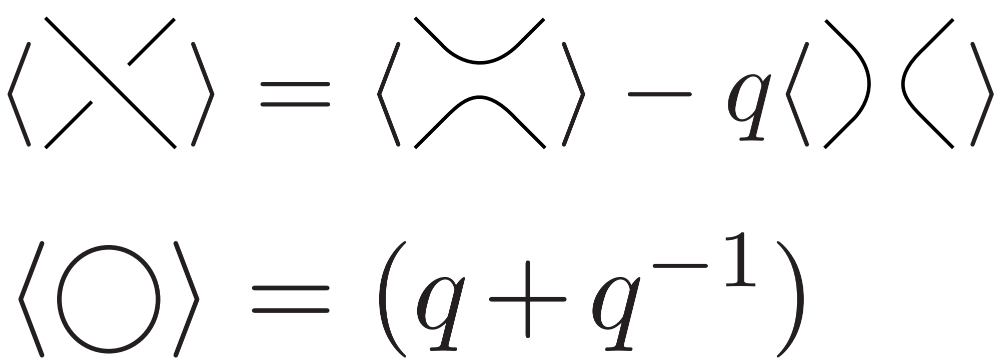


Reformulating the Bracket

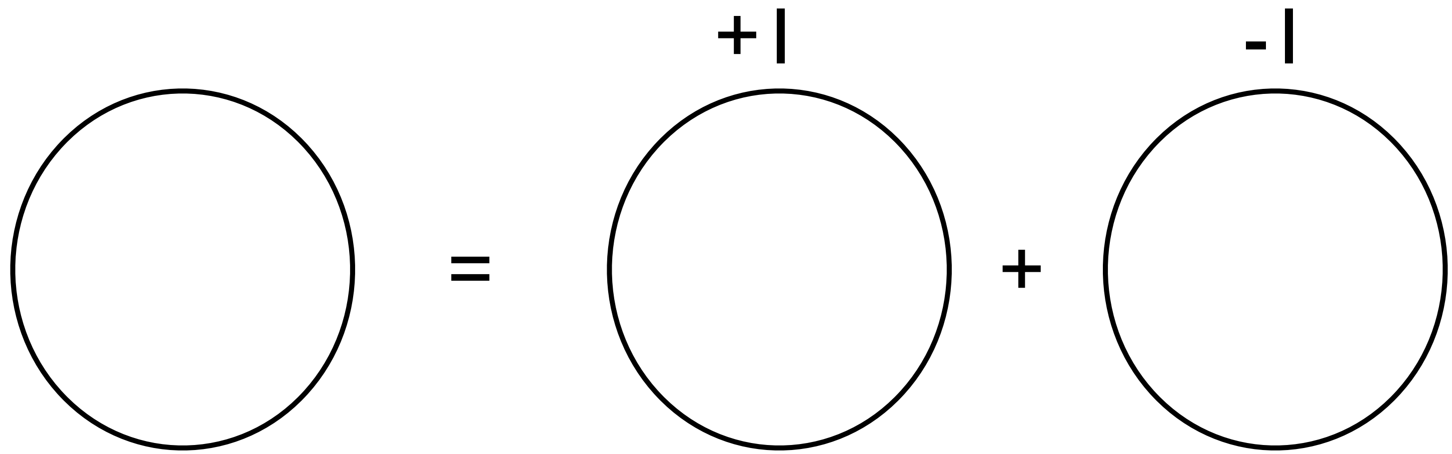
Let c(K) = number of crossings on link K.

Replace <K> by $A^{-c(K)}$<K> and replace A^{-2} by -q .

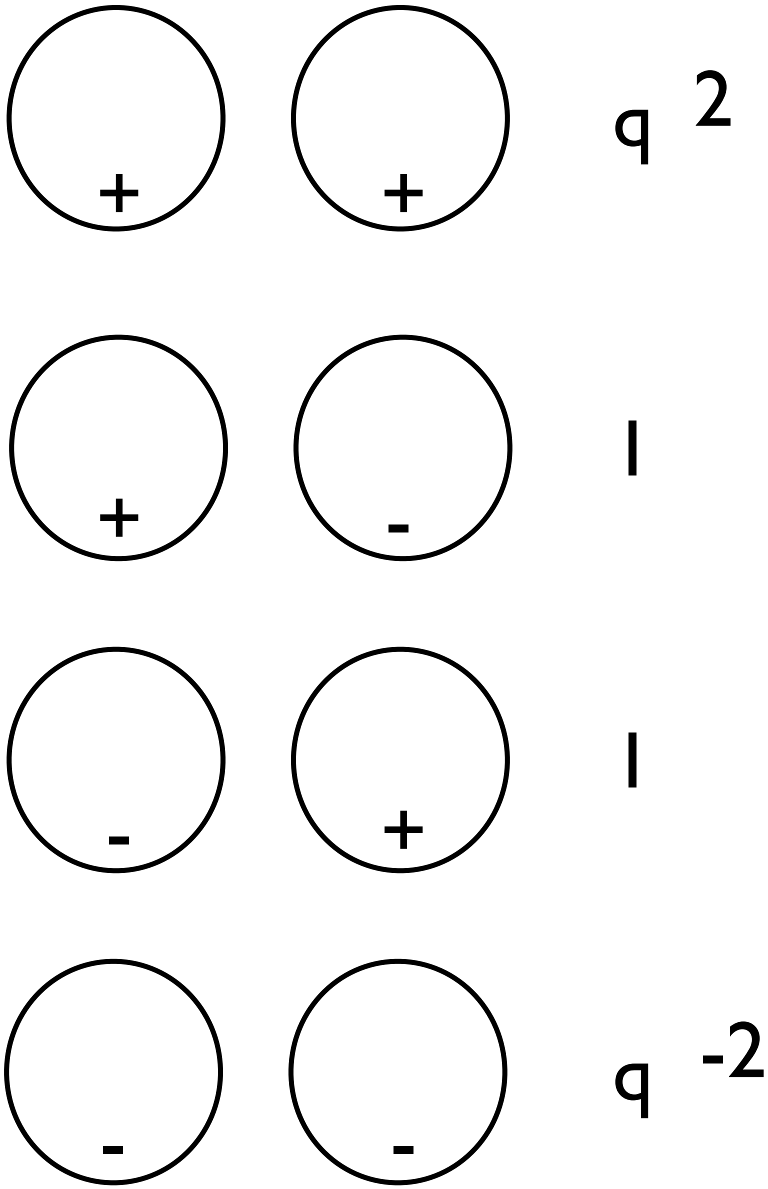
Then the skein relation for <K> will be replaced by:

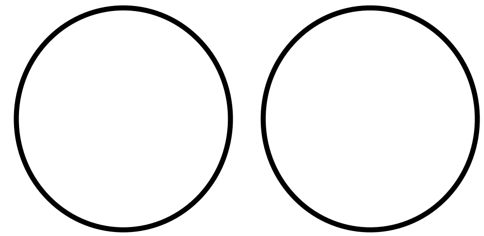


Use enhanced states (cf O.Viro)
by labeling each loop with +1 or -1.



$$\longleftrightarrow \quad q + q^{-1}$$

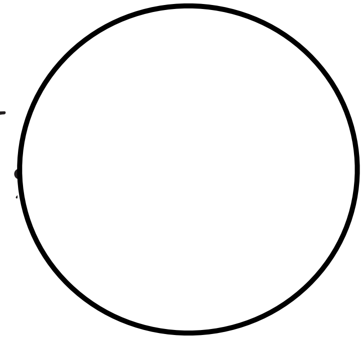




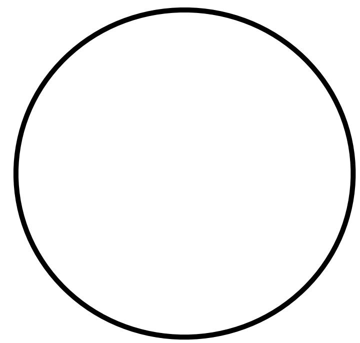
$(q + q^{-1})^2$

Enhanced States
circumvent the binomial
theorem.

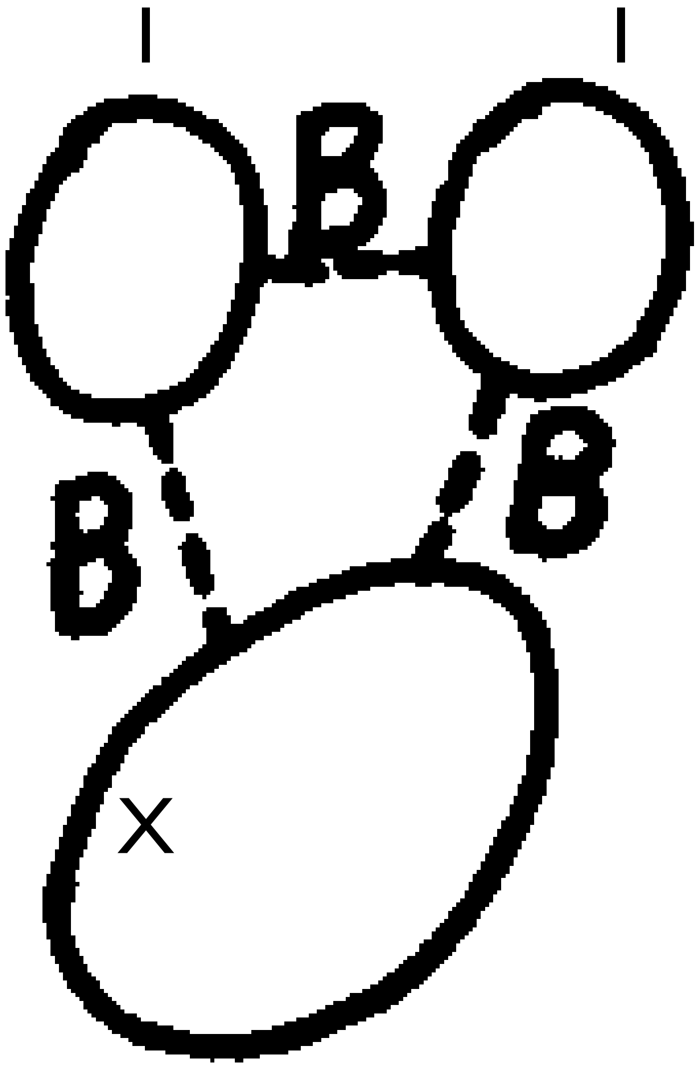
$$q^{-1} \Longleftrightarrow -1 \Longleftrightarrow X.$$



$$q^{+1} \Longleftrightarrow +1 \Longleftrightarrow 1$$



For reasons that will soon become apparent, we let (-1) be denoted by X and (+1) be denoted by 1.



An enhanced state
that contributes

[(q)(q)(1/q)] [(-q) (-q) (-q)]
1 1 X **B B B**

to the revised
bracket state sum.

Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_s q^{j(s)} (-1)^{i(s)}$$

Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_{s} q^{j(s)} (-1)^{i(s)}$$

$$j(s) = n_B(s) + \lambda(s)$$

$i(s) = n_B(s)$ = number of B-smoothings in the state s.

$\lambda(s)$ = number of +1 loops minus number of -1 loops.

(X)

$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

C^{ij} = module generated by enhanced states with $i = n_B$ and j as above.

Khovanov Homology - Jones Polynomial as a graded Euler Characteristic

$$\langle K \rangle = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet j}),$$

$$\langle K \rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j}))$$

A Quantum Digression

Let C(K) denote a Hilbert space with basis |s> where s runs over the enhanced states of a knot or link diagram K.

We define a unitary transformation.

$$U : \mathcal{C}(K) \longrightarrow \mathcal{C}(K)$$

$$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle$$

q is chosen on the unit circle in the complex plane.

$$|\psi\rangle = \sum_s |s\rangle$$

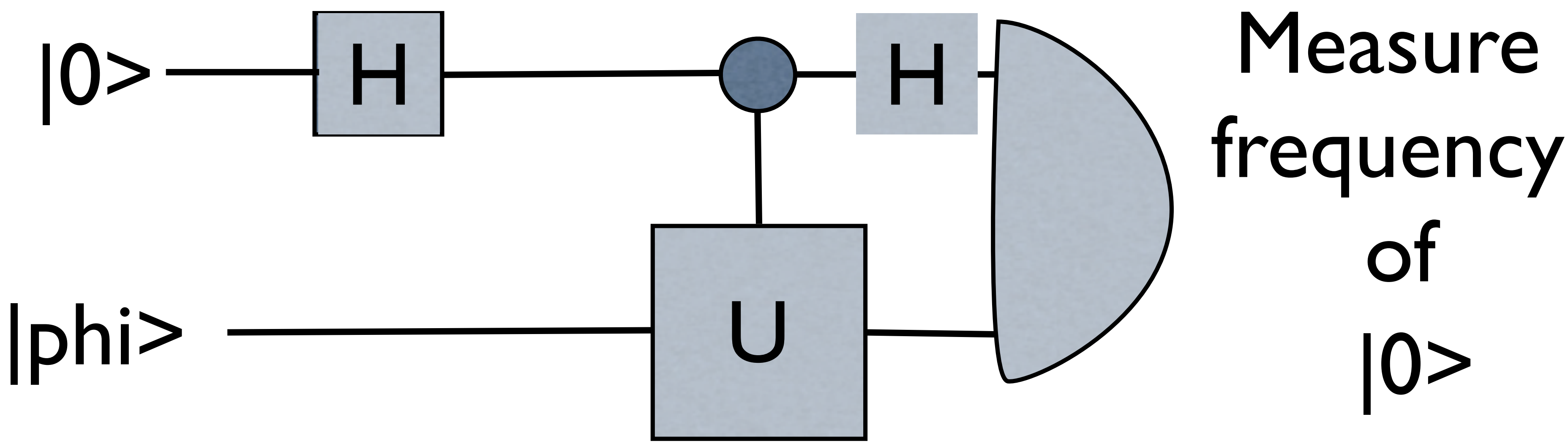
Lemma. The evaluation of the bracket polynomial is given by the following formula

<K> = Trace(U)

$$\langle K \rangle = \langle \psi | U | \psi \rangle .$$

This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).

Hadamard Test



$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} / \mathrm{Sqrt}(2)$$

|0> occurs with probability
1/2 + Re[<phi|U|phi>]/2.

Eigenspace Picture $\lambda = q^j$

$$\mathcal{C}^0 = \oplus_\lambda \mathcal{C}^0_\lambda$$

$$\mathcal{C}^\bullet_\lambda : \mathcal{C}^0_\lambda \longrightarrow \mathcal{C}^1_{-\lambda} \longrightarrow \mathcal{C}^2_{+\lambda} \longrightarrow \cdots \mathcal{C}^n_{(-1)^n\lambda}$$

$$\mathcal{C} = \oplus_\lambda \mathcal{C}^\bullet_\lambda$$

$$\langle\psi|U|\psi\rangle = \sum_\lambda \lambda\chi(H(C^\bullet_\lambda))$$

SUMMARY

We have interpreted the bracket polynomial as a quantum amplitude by making a Hilbert space C(K) whose basis is the collection of enhanced states of the bracket.

This space C(K) is naturally intepreted as the chain space for the Khovanov homology associated with the bracket polynomial.

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$

The homology and the unitary transformation U speak to one another via the formula

$$U\partial + \partial U = 0.$$

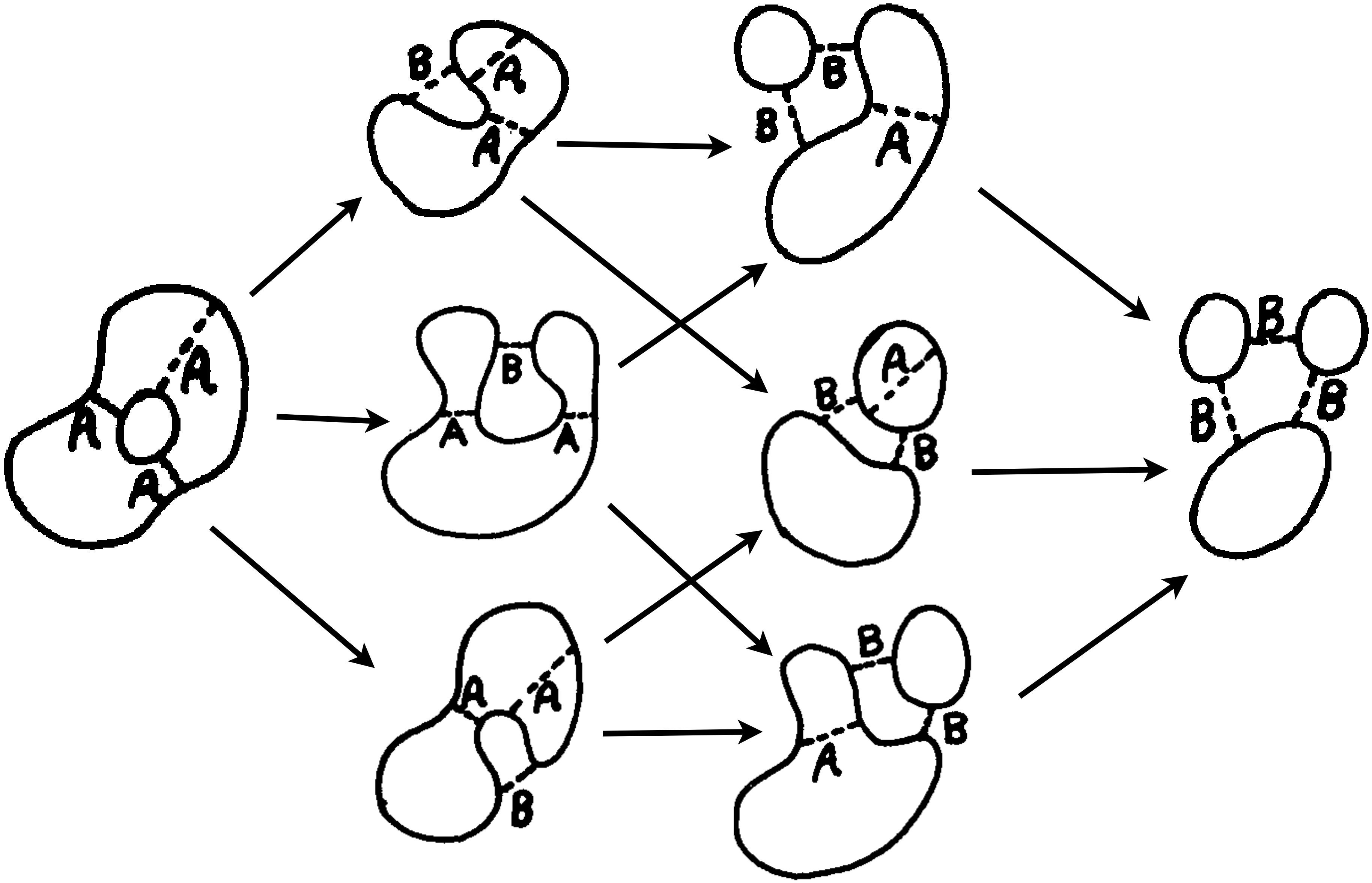
CATEGORIFICATION

View the next slide as a category whose objects are the bracket states.

The cubical shape of this category suggests making a homology theory.

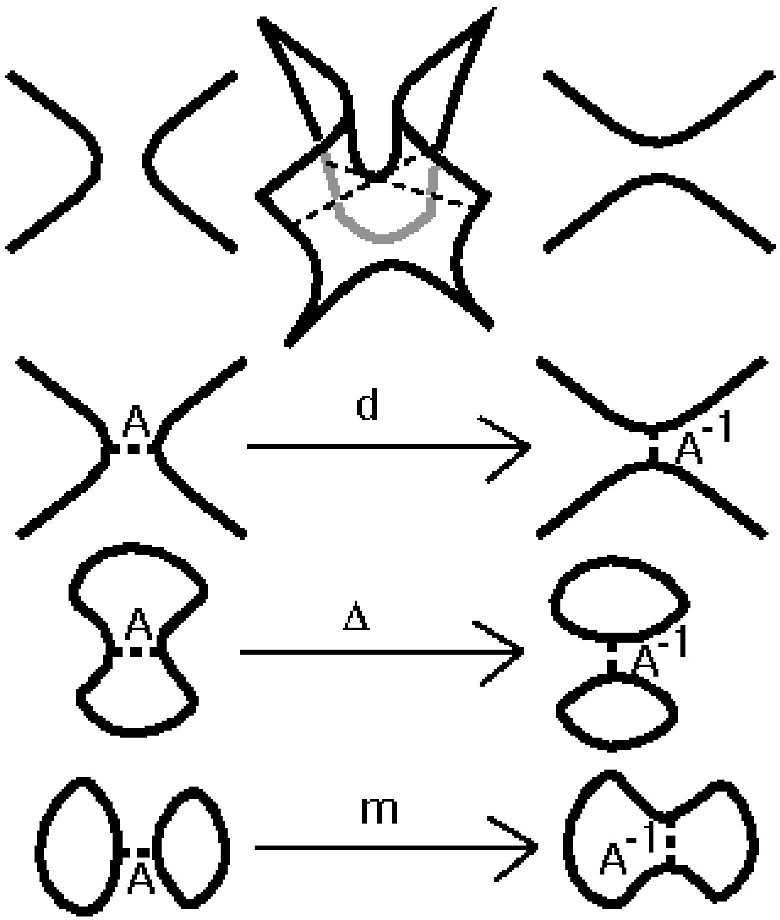
In order to make a non-trivial homology theory we need a functor from this category of states to a module category. Each state loop will map to a module V. Unions of loops will map to tenor products of this module.

The Khovanov Complex



$$\partial(s) = \sum_{\tau} \partial_{\tau}(s)$$

The boundary is a sum of partial differentials corresponding to resmoothings on the states.



Each state loop is a module.

A collection of state loops corresponds to a tensor product of these modules.

It turns out that one can take the algebra generated by 1 and X with
$X^2 = 0$ and

$$\Delta(X) = X \otimes X \text{ and } \Delta(1) = 1 \otimes X + X \otimes 1.$$

The chain complex is then generated by enhanced states with loop labels 1 and X.

$$\langle K \rangle = \sum_{i,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

Khovanov differential acts in the form

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

For j to be constant as i increases by 1, we need

$\lambda(s)$ to decrease by 1.

[which it does!]

The differential increases the homological grading i by 1 and leaves fixed the quantum grading j.

Then

$$\langle K\rangle = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet j})$$

$$\chi(H(\mathcal{C}^{\bullet j})) = \chi(\mathcal{C}^{\bullet j})$$

$$\langle K\rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j}))$$

ARROW POLYNOMIAL

The arrow polynomial is a generalization of the Jones polynomial (bracket polynomial) that takes into account the state structure of oriented diagrams.

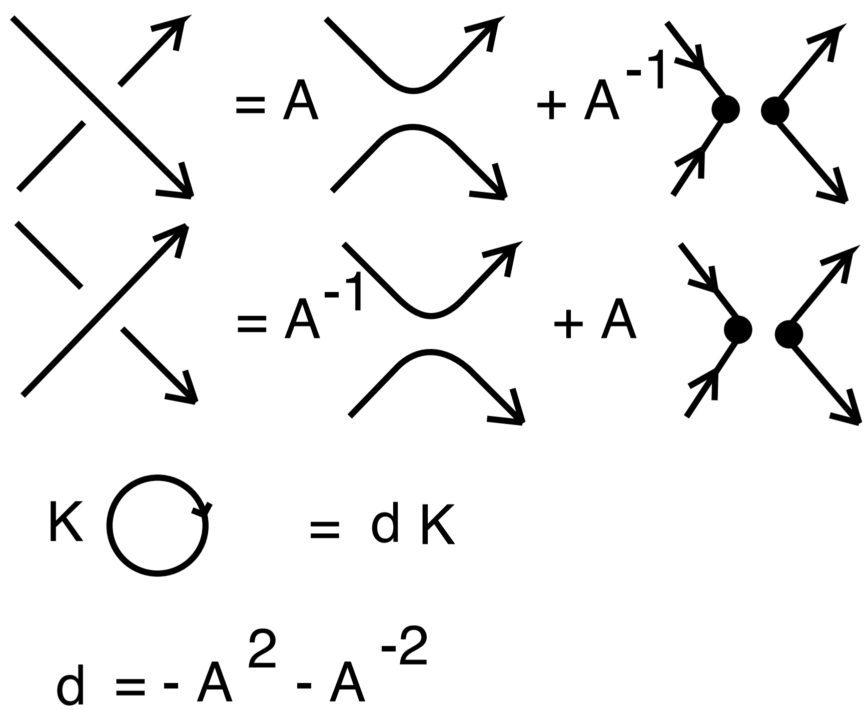
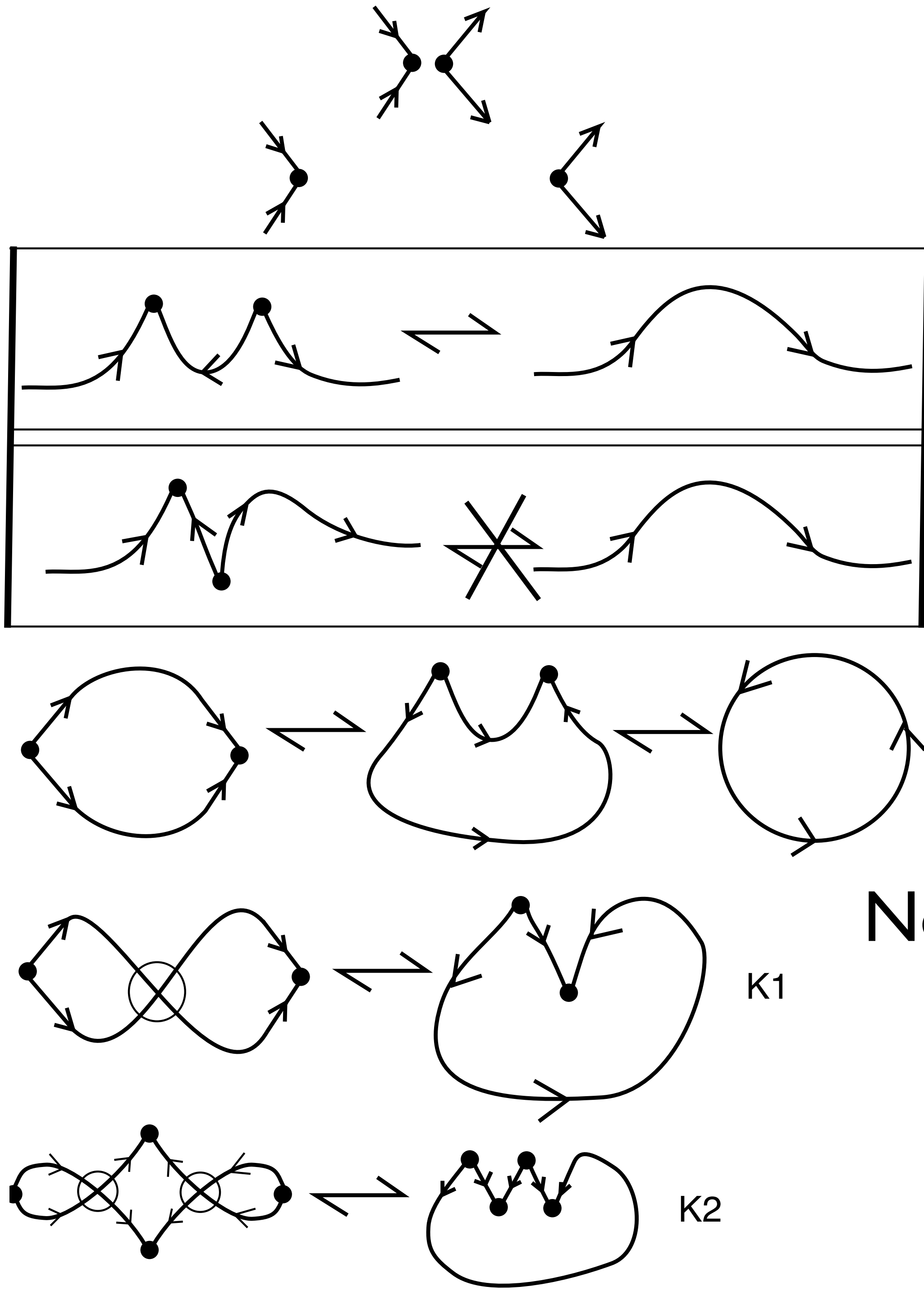
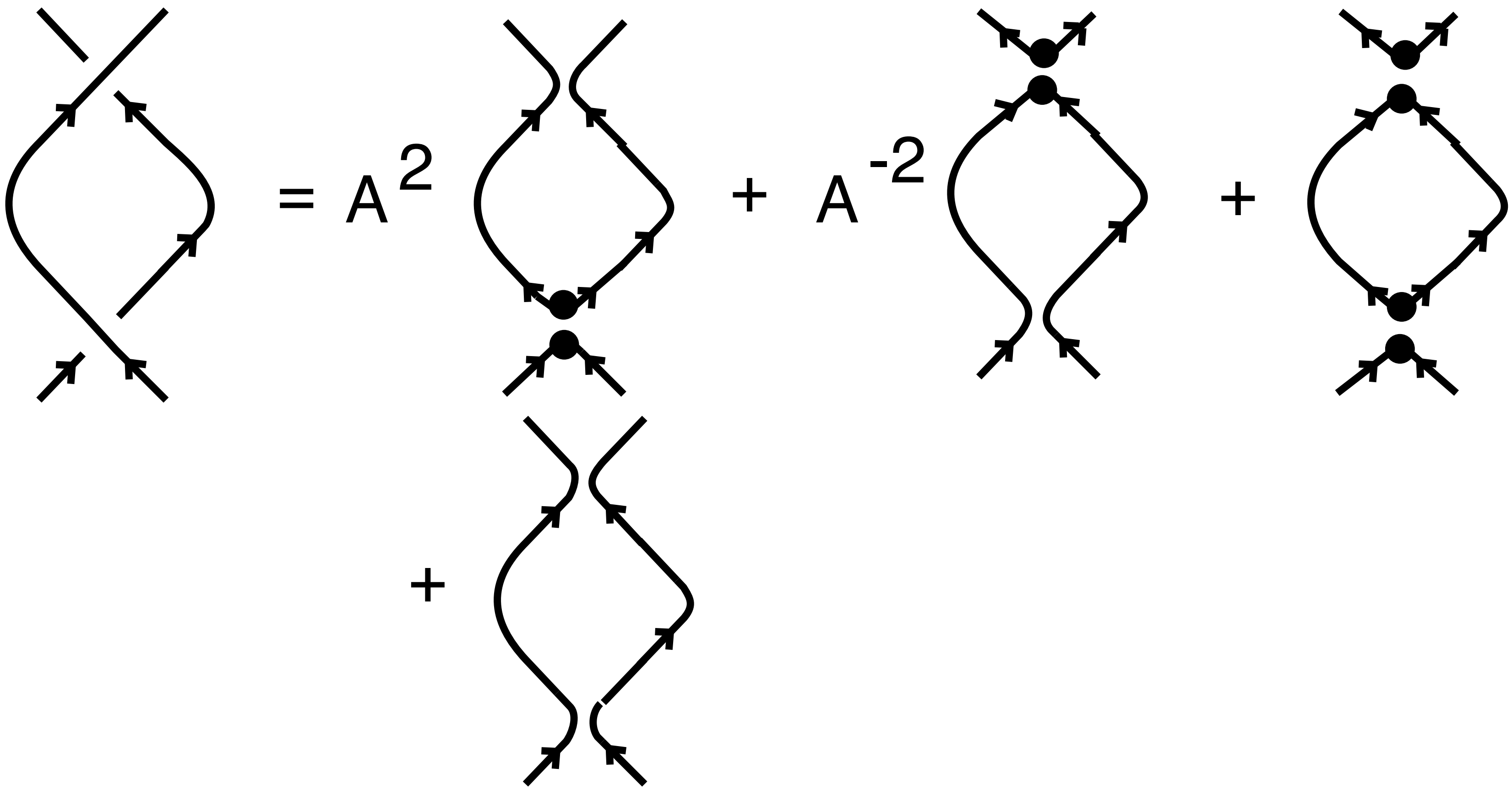


Figure 1: **Oriented Bracket Expansion.**

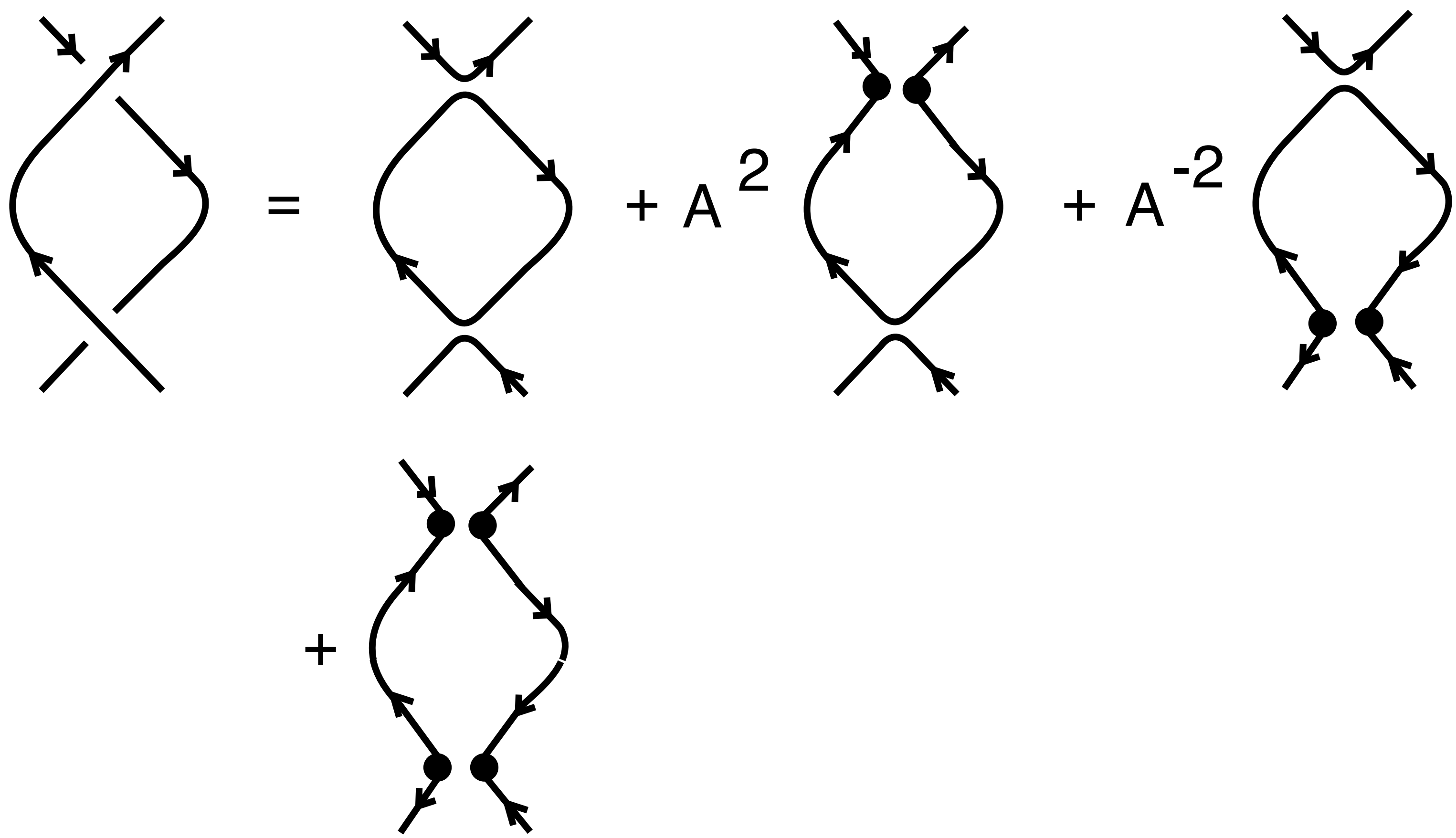


Sufficient for
invariance under
Reidemeister moves

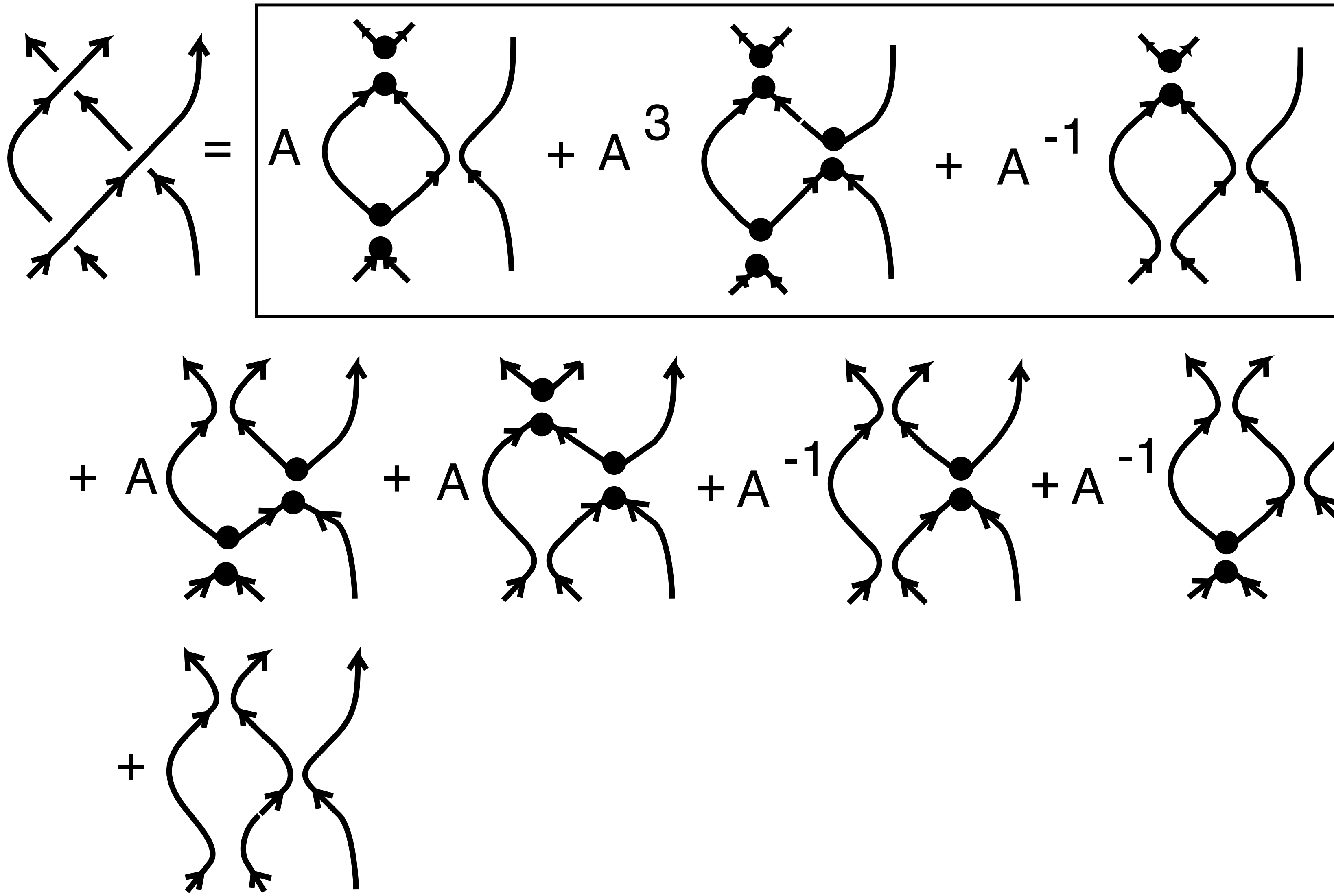
Non trivial reduced loops
do not occur
for classical knots.



Oriented Second Reidemeister Move

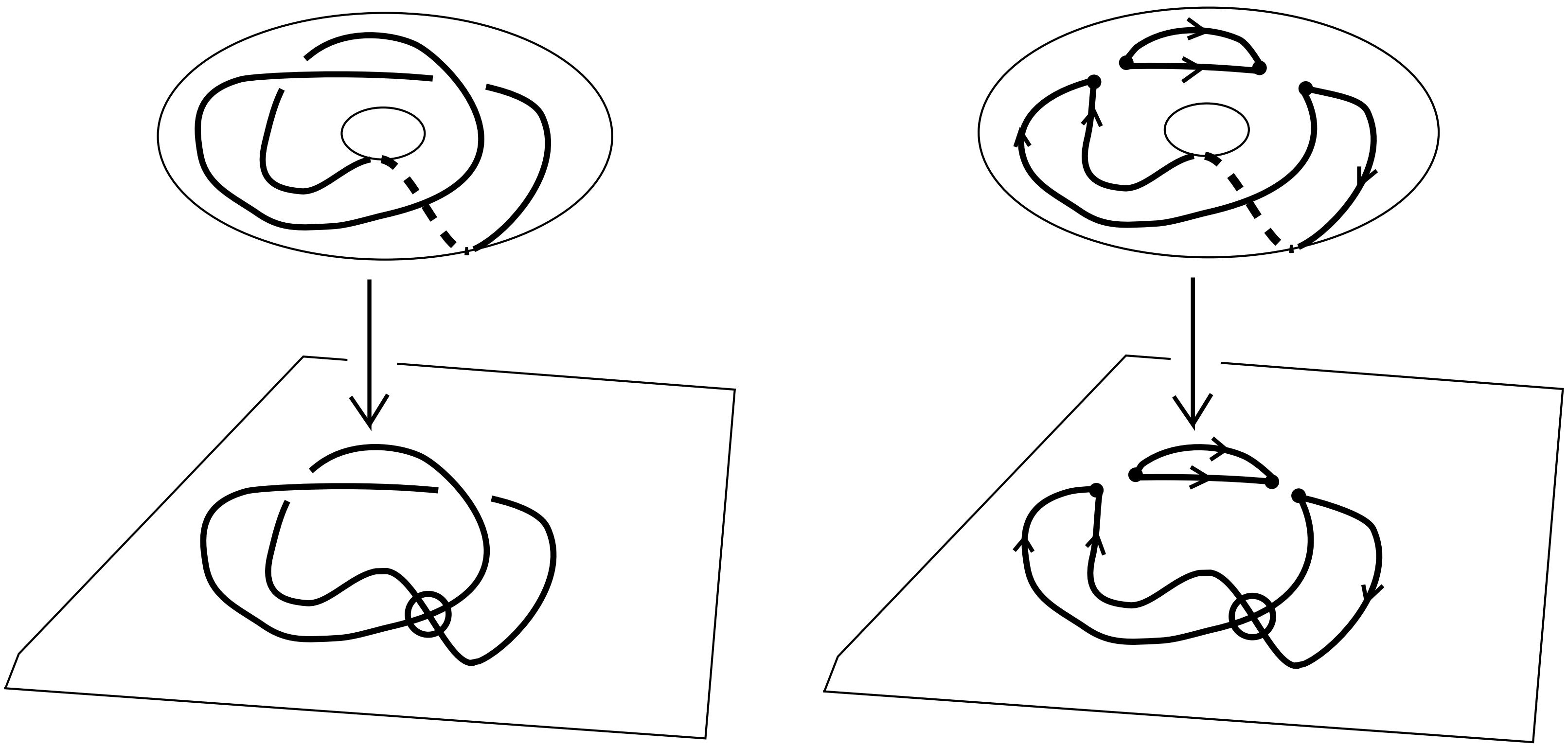


Reverse Oriented Second
Reidemeister Move



Third Reidemeister Move

For a knot in a thickened surface, the states can zigzag and so have arrow numbers.



Virtual Knot Theory
studies stabilized knots in thickened surfaces.

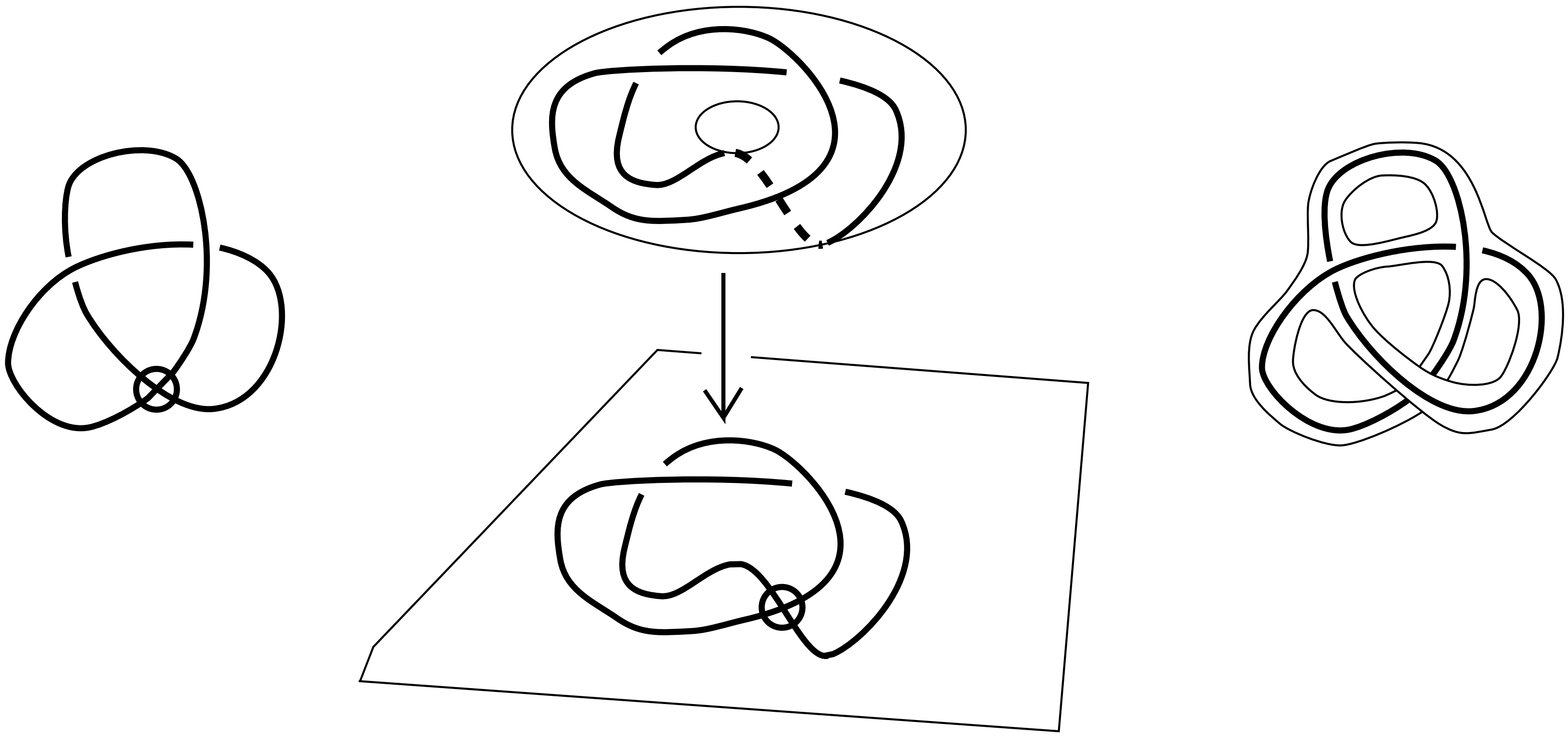
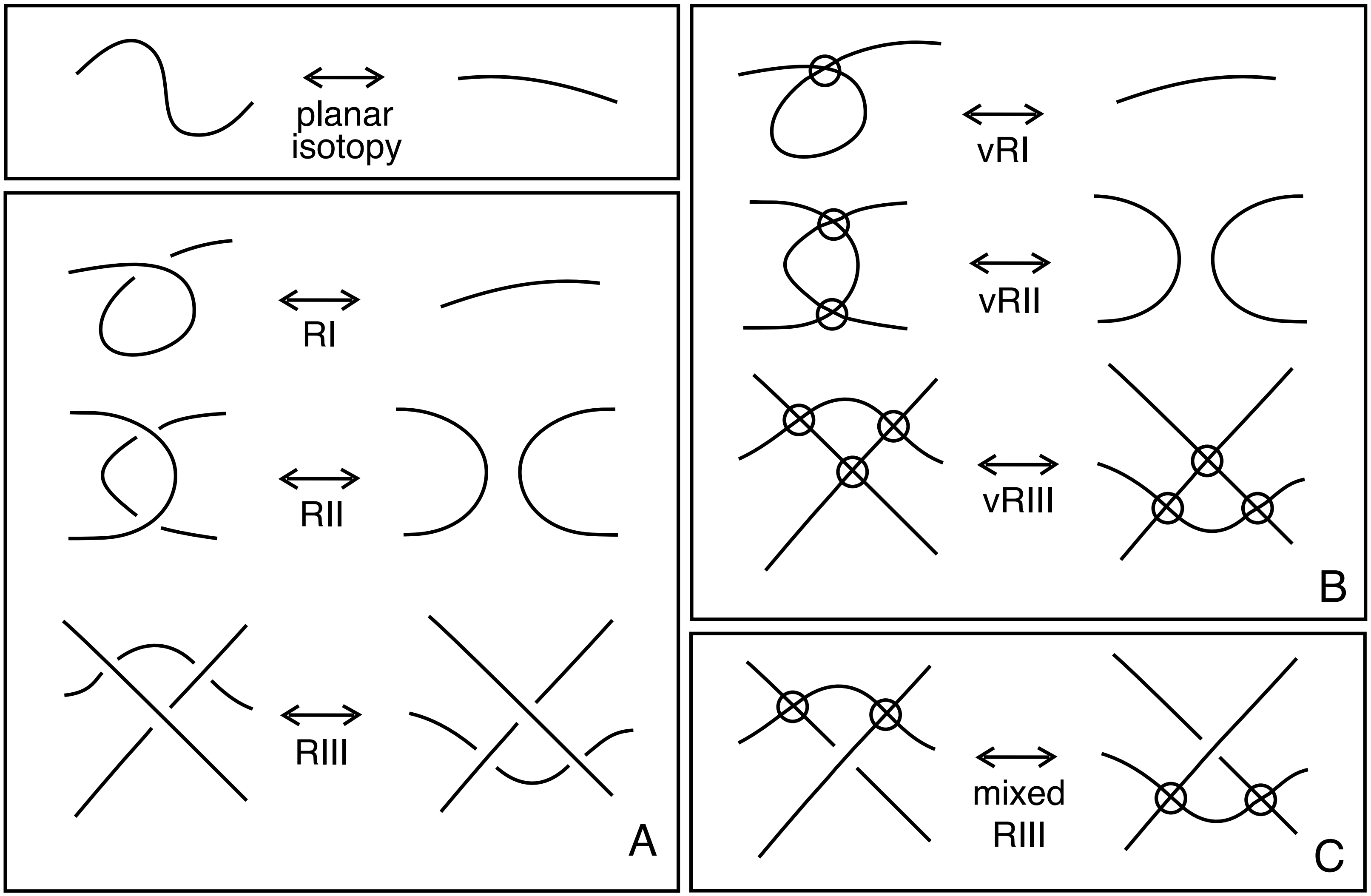


Figure 4: **Surfaces and Virtuals**

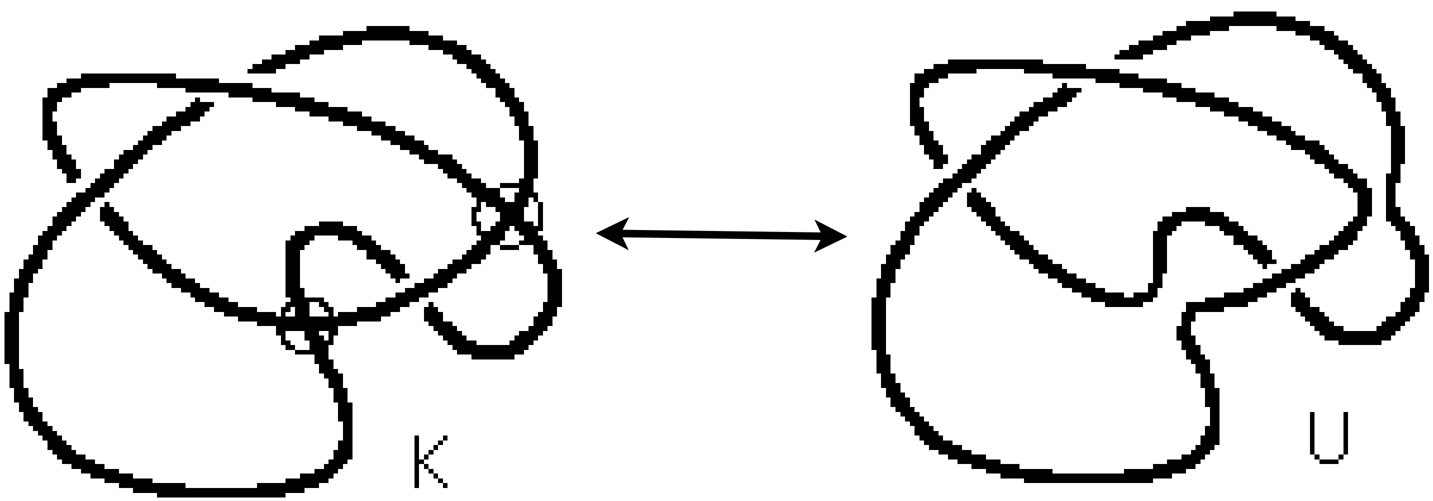
Generalized Reidemeister Moves for Virtual Knots and Links



There exist infinitely many non-trivial K
with unit Jones polynomial.

Bracket Polynomial is Unchanged
when smoothing flanking virtuals.

Z-Equivalence



Classical knot theory embeds in virtual knot theory.

Open Question:
Does classical knot theory embed in virtual knot theory modulo Z-equivalence?

Theorem 1. *With the above conventions, the arrow polynomial $\langle K \rangle_A$ is a polynomial in A, A^{-1} and the graphical variables K_n (of which finitely many will appear for any given virtual knot or link). $\langle K \rangle_A$ is a regular isotopy invariant of virtual knots and links. The normalized version*

$$W[K] = (-A^3)^{-wr(K)} \langle K \rangle_A$$

is an invariant of virtual isotopy. Here $wr(K)$ denotes the writhe of the diagram K; this is the sum of the signs of all the classical crossings in the diagram. If we set $A = 1$ and $d = -A^2 - A^{-2} = -2$, then the resulting specialization

$$F[K] = \langle K \rangle_A (A = 1)$$

is an invariant of flat virtual knots and links.

Example. Figure 4 illustrates the Kishino diagram. With $d = -A^2 - A^{-2}$

$$\langle K \rangle_A = 1 + A^4 + A^{-4} - d^2 K_1^2 + 2K_2.$$

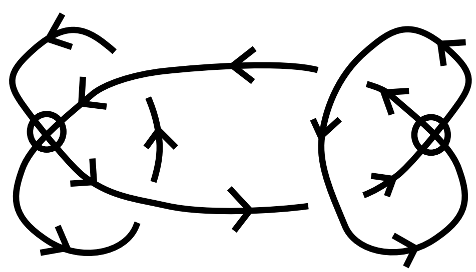
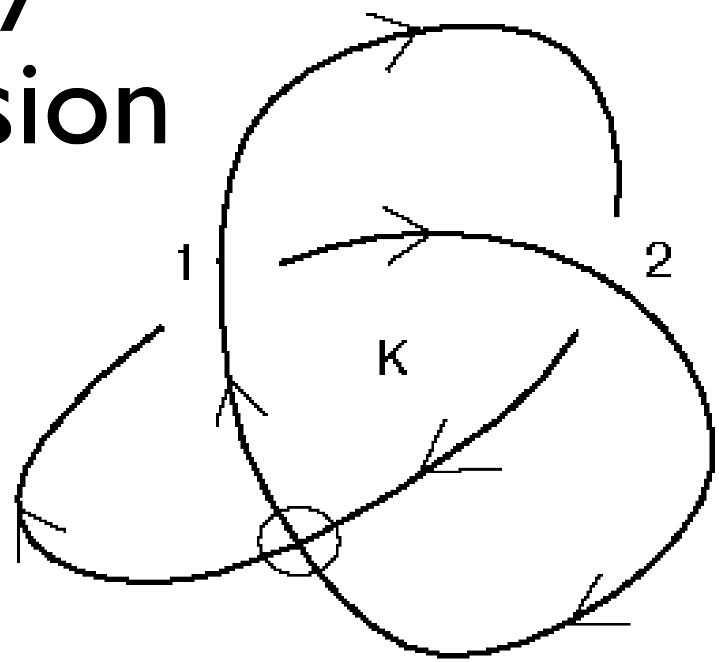


Figure 4: **Kishino Diagram.**

Parity Digression

The Odd Writhe



Bare Gauss Code
1212

Crossings 1 and 2 are odd.

A crossing is odd if it flanks an odd number of symbols in the Gauss code.

The odd writhe of K, J(K).

J(K) = Sum of signs of the odd crossings of K.

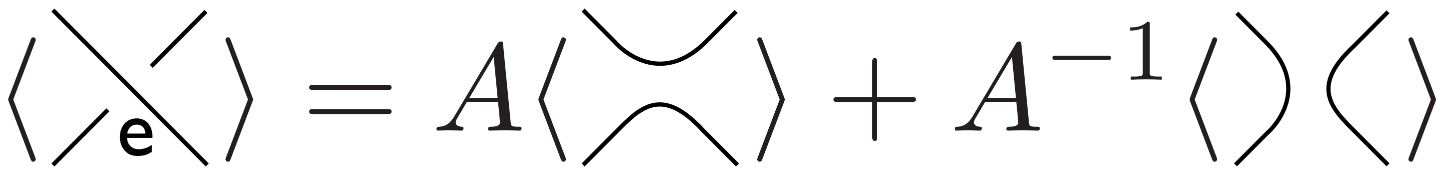
Here J(K) = -2.

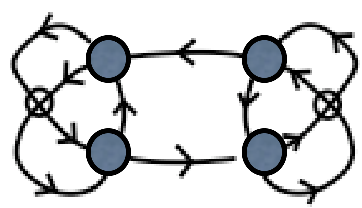
Facts: J(K) is an invariant of virtual isotopy.
J(K) = 0 is K is classical.
J(Mirror Image of K) = -J(K).

Hence this example is not classical and is not isotopic to its mirror image.

Parity Digression

Manturov's Parity Bracket

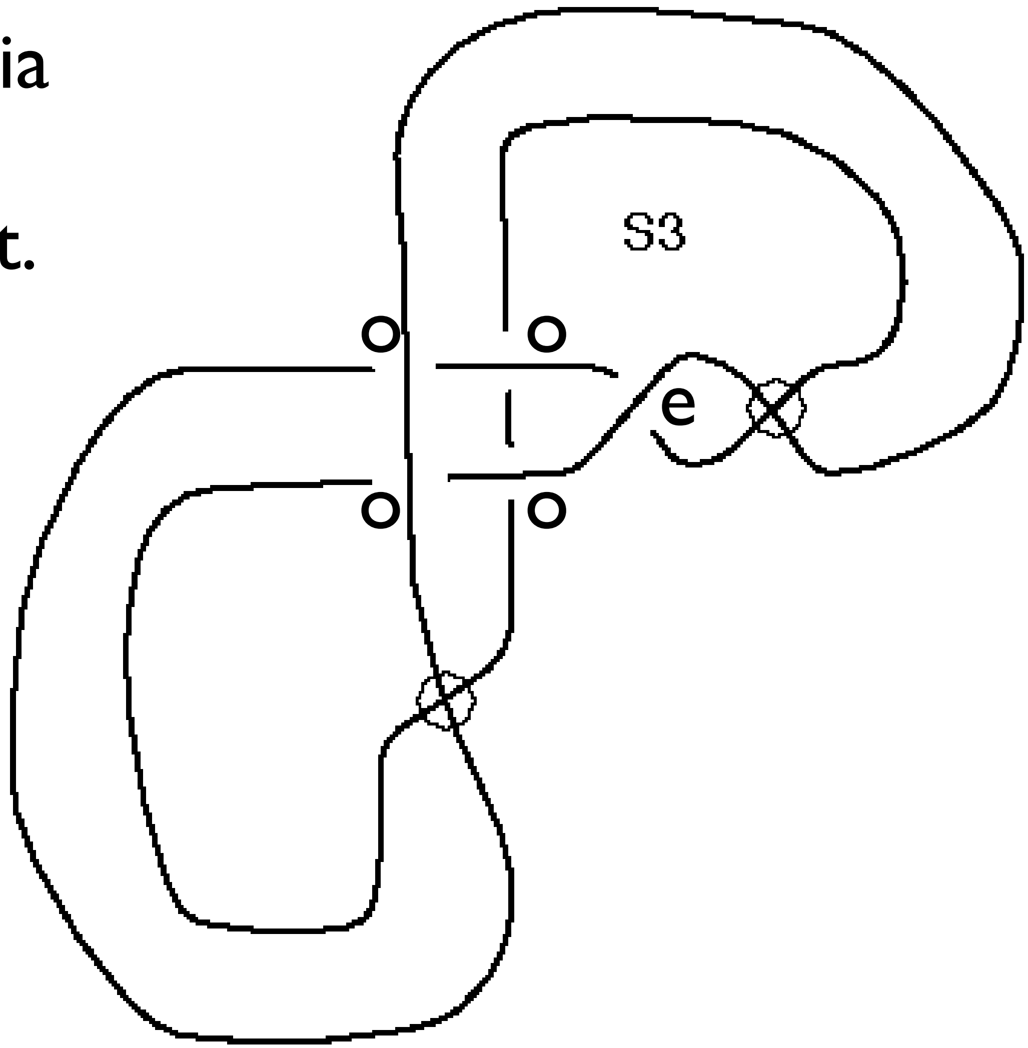




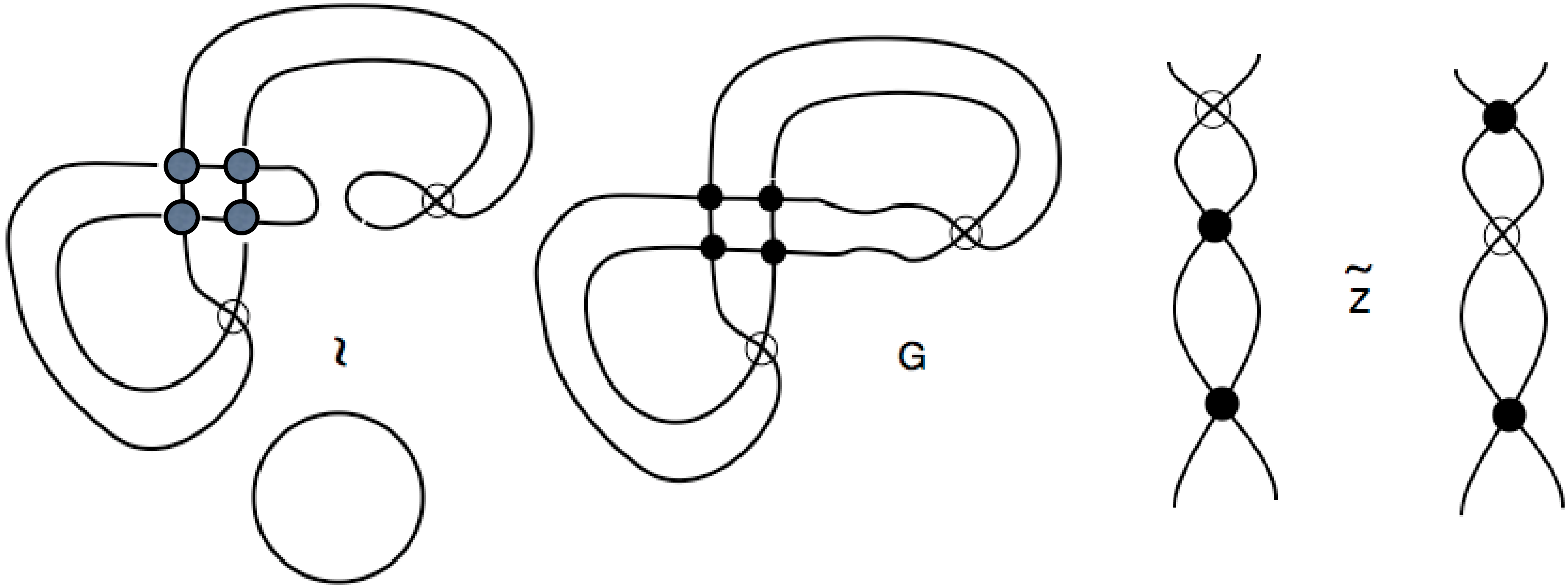
The Parity Bracket provides the simplest proof that the Kishino diagram is non-trivial.

The Knot S3 (found with Slavik Jablan) has unit Jones polynomial. It is not Z-equivalent to a classical knot.

Proof via Parity Bracket.

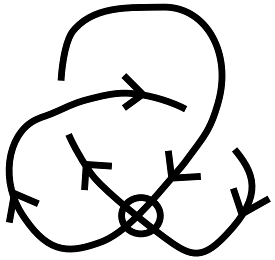


A[S3] = -2K1^2 + K2 + A^4 (1 - 2K1^2 + K2)



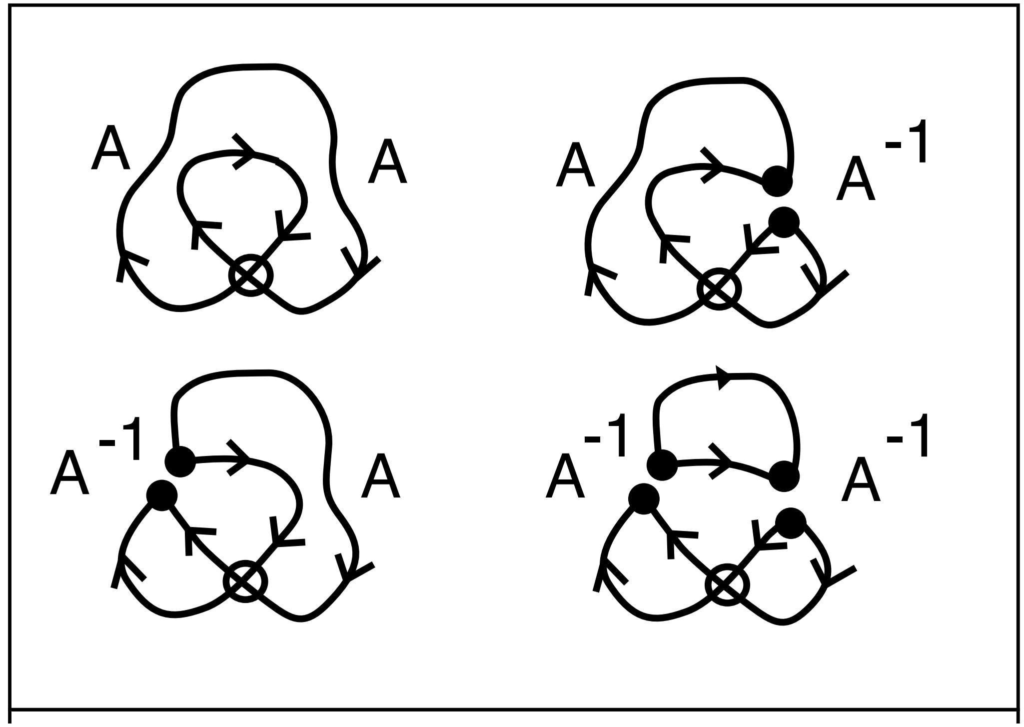
The Parity bracket of S3 has only two terms and includes the graph G. The virtual graph G cannot be reduced by Reidemeister Two moves on its nodes.

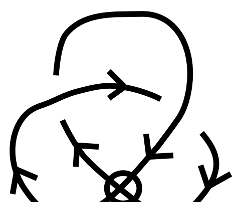
More Arrow Computation



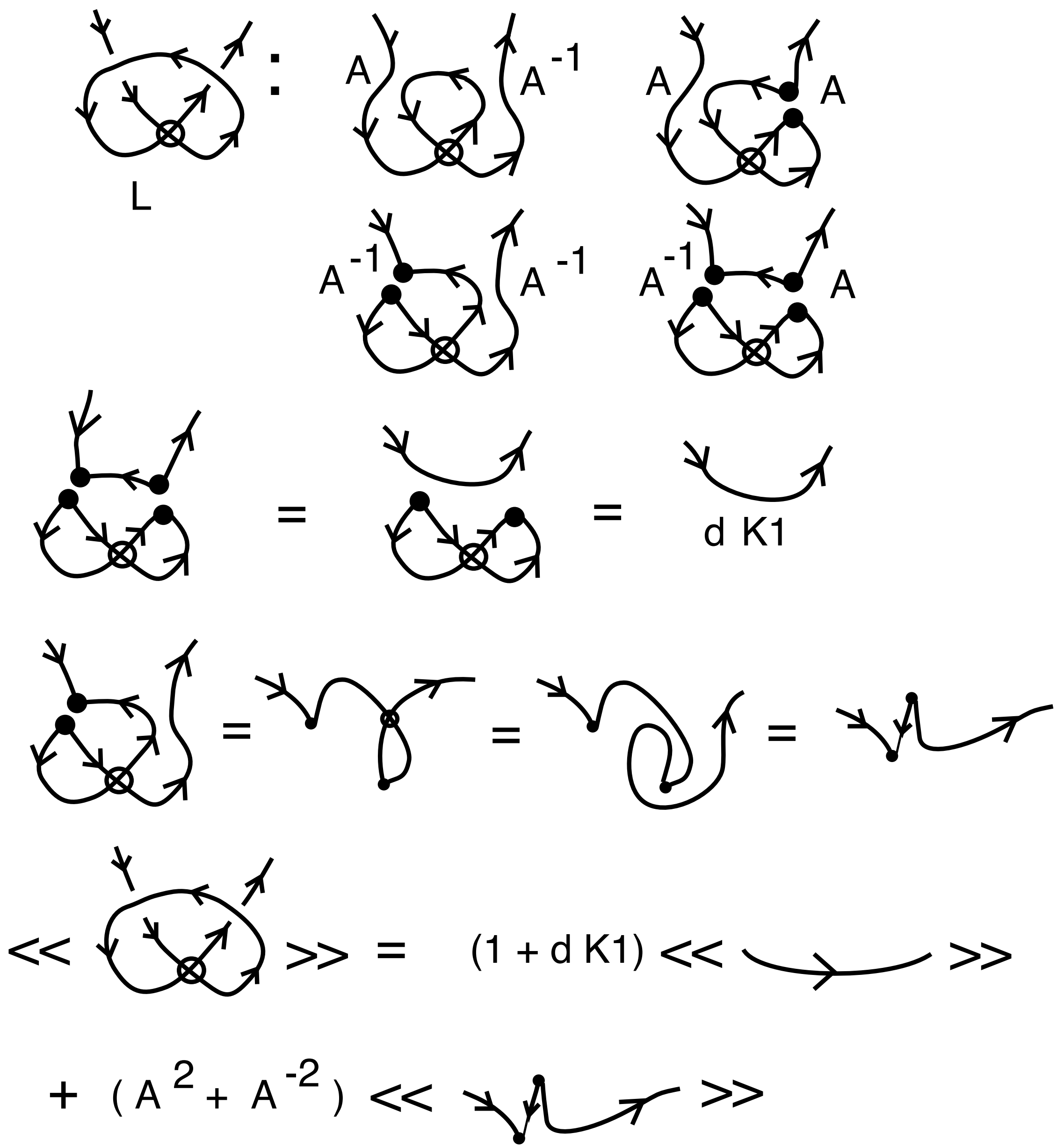
K

:



<<  >> = $A^2 + (2 + A^{-2} d)$ K1

Arrow Polynomial Can Detect Long Knots whose closures are trivial.



Heading Toward Categorification of Arrow States

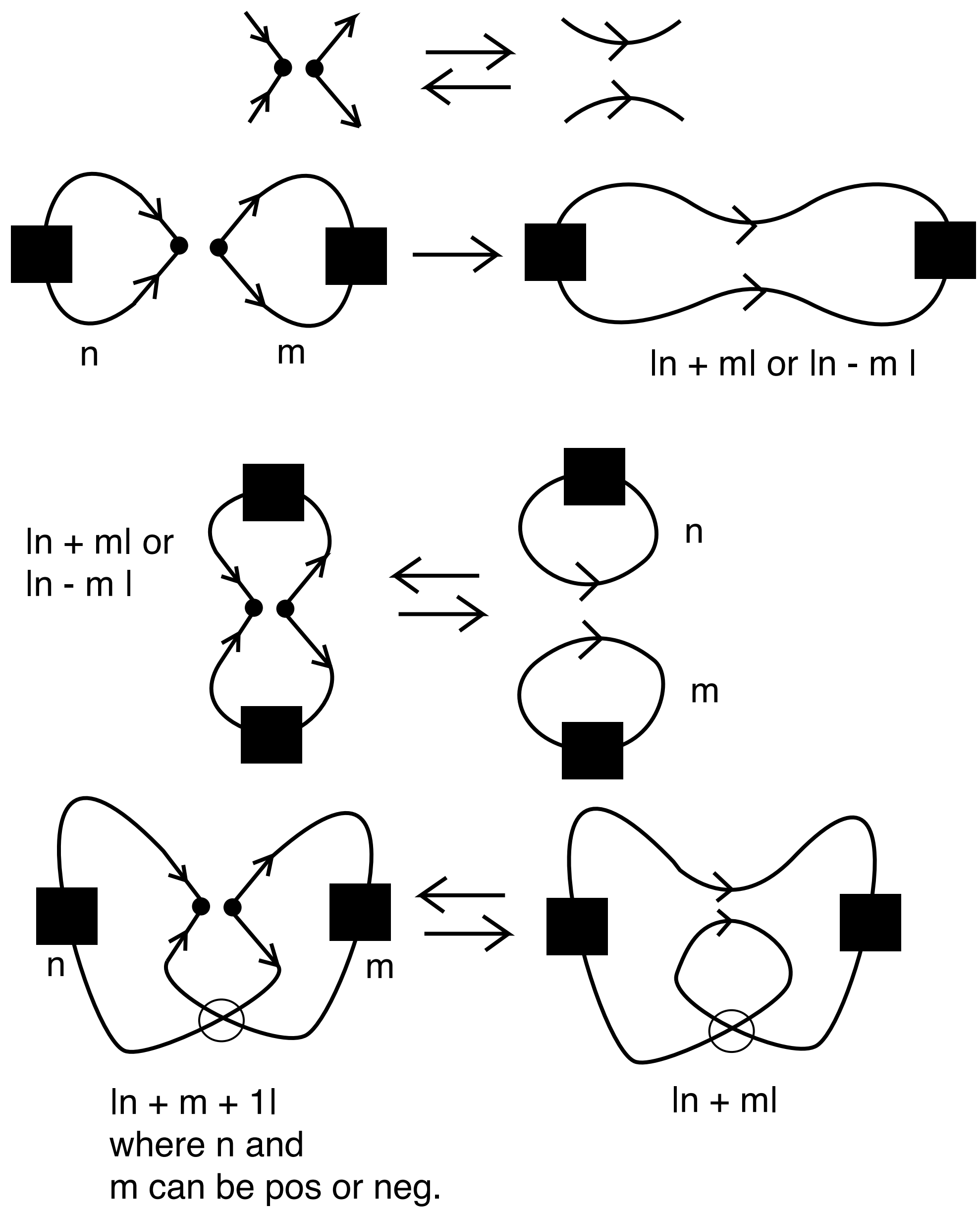


Figure 6: **Arrow Numbers for Interacting Loops**

Dotted Gradings and Dotted Categorification

Assume there is some method for assigning "dots" to some state circles such that the following conditions hold.

1. The dotting of circles is additive with respect to $2 \to 1$-bifurcations and $1 \to 2$-bifurcations mod 2. This additivity means that when we merge two circles (split one circle into two), the number of dots on the circles being operated on is preserved modulo $\mathbf{Z}_2$.

   This means that the parity of the number of dots on the circles operated on is preserved whenever we merge two circles or split one circle into two.

   If the dotting is not preserved under a $1 \to 1$ bifurcation, then this bifurcation is taken to be the zero map.
2. Similar curves for corresponding smoothings of the RHS and the LHS of any Reidemeister move have the same dotting.
3. Small circles appearing for the first, the second, and the third Reidemeister moves are not dotted.

Example: Count the arrow number modulo two.

Let us call the conditions above *the dotting conditions*. With such a structure in hand, one defines a *new grading* $g(s)$ for states s by taking the difference between the number of dotted X's and the number of dotted 1's in the state.

$$g(s) = \sharp(\dot{X}) - \sharp(\dot{1})$$

Theorem 3. *Assume there is a theory using the Khovanov complex $([[K]], \partial)$ such that the Kauffman states can be dotted so that the dotting conditions hold. Take $[[K]]_g$ to be the space $[[K]]$ endowed with new grading as above.*

Define ∂' to be the composition of ∂ with the new grading projection and set $\partial'' = \partial - \partial'$.

Then the homology of $[[K]]_g$ (with respect to ∂') is invariant (up to a degree shift and a height shift).

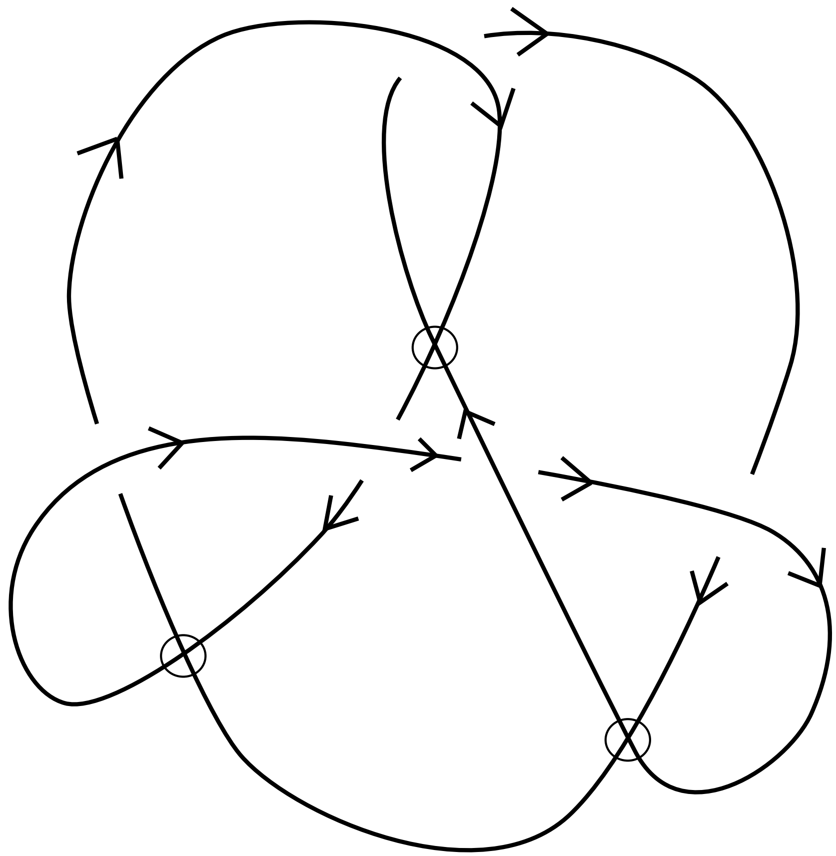
Now, one can easily check that the conditions of the theorem hold if we set the dotting as follows: the curve is dotted if it is marked as K_j with j odd, and it is not dotted if it is marked as K_i with i even.

Kaestner's example of a pair of virtual knots with the same arrow polynomial but distinct categorification homology.

These two knots are NOT distinguished by mod-2 Khovanov homology or by the arrow polynomial.

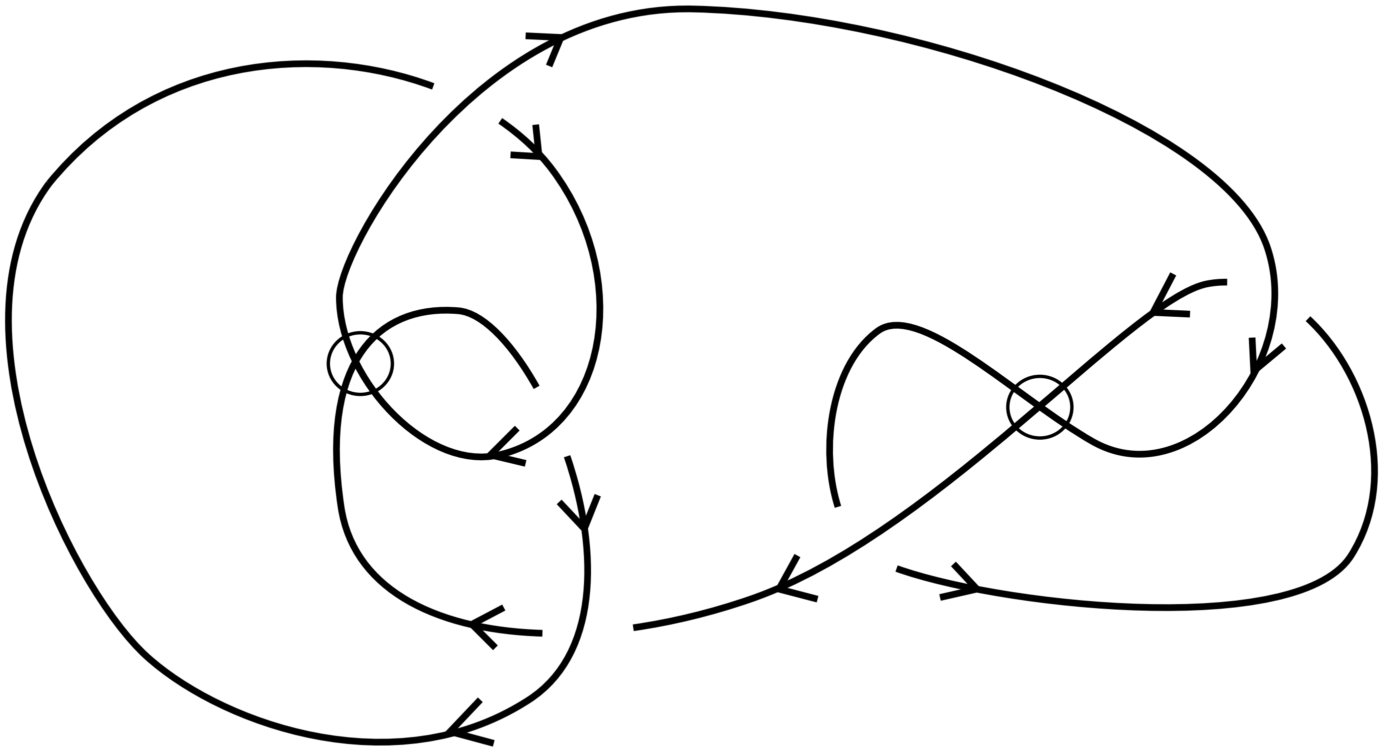
They ARE distinguished by the first (mod-2 arrow number dotting) categorification of the arrow polynomial.

VK5[267]



$$1+\frac{2}{q^3}+\frac{1}{q^2}+\frac{4}{q}+2q+\frac{1}{q^5t^2}+\frac{1}{q^3t^2}+\frac{3}{q^3t}+\frac{3}{qt}+t+\frac{t}{q}+qt+q^2t$$

VK5[129]



$$1+\frac{2}{q^3}+\frac{1}{q^2}+\frac{4}{q}+2q+\frac{1}{q^5 t^2}+\frac{1}{q^3 t^2}+\frac{3}{q^3 t}+\frac{3}{q t}+t+\frac{t}{q}+q t+q^2 t$$

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AKh[GCtoOrientedPD[VKFive[129]]] - AKh[GCtoOrientedPD[VKFive[267]]]
```

$$\frac{1}{q}+q+\frac{t}{q}+q t$$

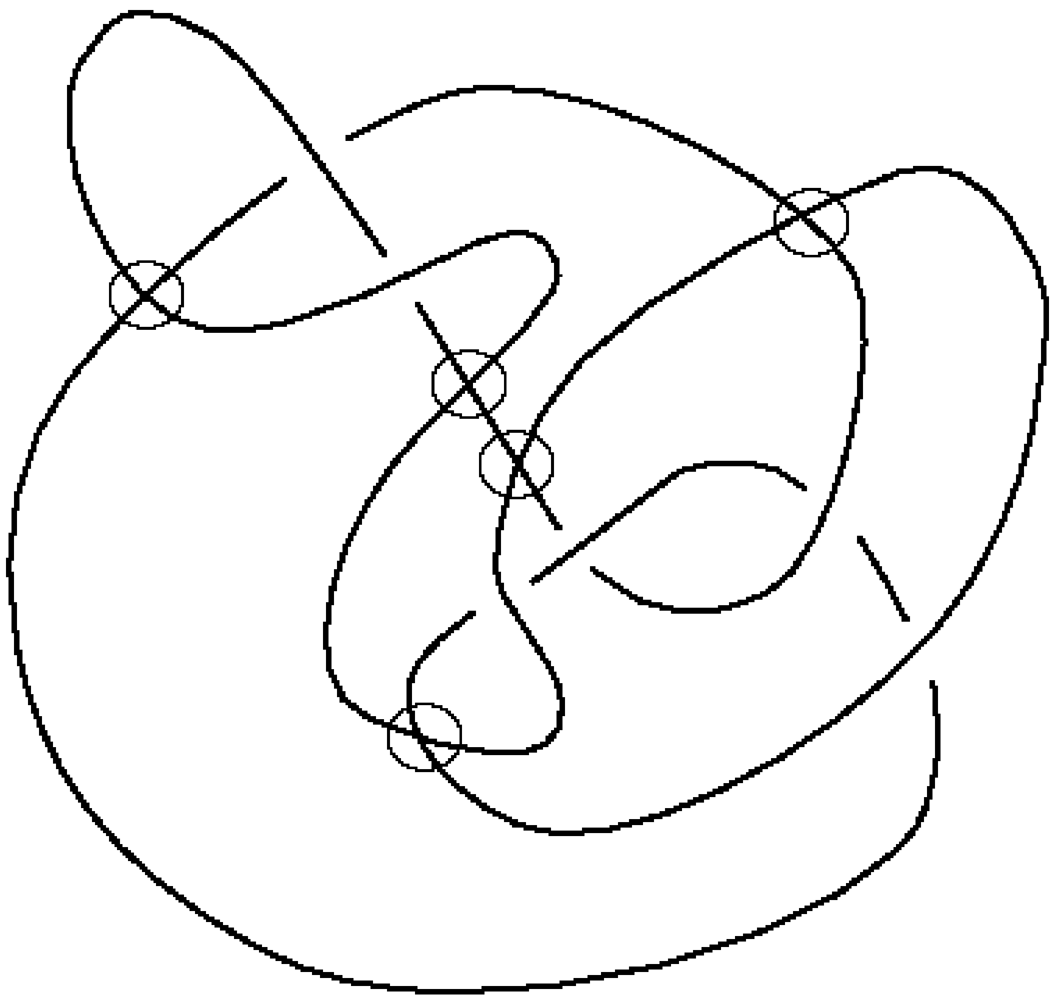
In Jeremy Green's tables there are 2448 five crossing virtual knots (tabulated).
There are 28 sets of knots with same arrow poly and some different categorifications within each set.

The two knots in the previous two slides each have arrow polynomial:

$$2A - A^5 - \frac{K1}{A^5} + \frac{K1}{A} - \frac{2K1^2}{A^3} - 2AK1^2 + \frac{K2}{A^3} + AK2$$

Note that VK5[267] appears with three virtual crossings.
The arrow polynomial says that the virtual crossing number is at least two. Is it three?

The following example is not detected by any of our invariants.



VKSix[32008]

MANY QUESTIONS

Are there knots unseen by Arrow, but seen by ArrowCat?

What is going on in the examples we have found?

What does adding parity to Arrow and ArrowCat yield?
(more on this soon in paper of LK and AK)

How does all this reflect on the problem whether bracket/Jones poly detects unknot?

How does ArrowCat information reflect on the fact that Khovanov Homology detects the unknot?