The search for gravity waves has proven to be a daunting task for experimental physicists, pushing the limits of current technology. However, researchers now appear to at the frontier, and within the next few years, gravity waves may be discovered for the first time. There is good reason to be excited: this discovery may expand our view of the universe many times over, and a new form of astronomy may be born. Many even speak of a scientific revolution. This paper is intended to give a whirlwind tour of both the theory and practice of gravitational wave detection. Essentially, the first half will be dedicated to the mathematical basis of gravitational waves, and how they are predicted by general relativity. The second half will look at the more practical issues of detector design and noise reduction techniques.

THE NATURE OF GRAVITATIONAL WAVES

Newton's gravitational theory works well in the weak fields encountered in our patch of the universe (i.e. the solar system). However, it quietly glosses over the mechanism of gravity, simply stating "action at a distance." Presumably, then, if a mass were to move quickly from one region of space to another, this change would be noted instantaneously throughout the entire universe. But this is in direct conflict with special relativity. As will be shown, gravitational waves are the relativistic response to this problem.

Unlike mechanical or electromagnetic waves, gravity waves are non-linear, meaning (amongst other things) that the principle of superposition[1] does not apply. Fortunately for us, in areas where the gravitational field is weak, a first order approximation is appropriate, and linearity can be retrieved. In terms of general relativity, regions of spacetime with weak gravity can be considered

[1] Richard Feynman thought the principle of superposition was one of the luckiest results in physics. Certainly, it need not apply.

"nearly" flat. Thus, in a linear approximation, the metric $g_{\mu\nu}$ characterizing the space may be expressed as the flat Minkowski metric plus a perturbation term,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1 \tag{1}$$

where $\eta_{\mu\nu}$ = diag(-1,+1,+1,+1) is in canonical form. We shall assume $h_{\mu\nu}$ is so small that higher order terms are negligible. Thus:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{2}$$

Note that since $g_{\mu\nu}$ and $\eta_{\mu\nu}$ are both symmetric, it follows that $h_{\mu\nu}$ (and thus $h^{\mu\nu}$) is also symmetric. As well, it is clear that changes in $g_{\mu\nu}$ will be the result of $h_{\mu\nu}$ only; therefore, our ultimate goal will be to create a linearized filed equation in terms of $h_{\mu\nu}$. To this end, we start by calculating the Christoffel symbols:

$$\begin{aligned}\Gamma^{\rho}{}_{\mu\nu} &= \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\ &= \frac{1}{2}(\eta^{\rho\lambda} - h^{\rho\lambda})[\partial_\mu(\eta_{\nu\lambda} + h_{\nu\lambda}) + \partial_\nu(\eta_{\lambda\mu} + h_{\lambda\mu}) - \partial_\lambda(\eta_{\mu\nu} + h_{\mu\nu})]\end{aligned} \tag{3}$$

Expanding this out, we note any partial derivative acting on $\eta_{\mu\nu}$ will be zero ($\eta_{\mu\nu}$ is constant) and higher order terms of $h_{\mu\nu}$ will be removed to maintain a first order approximation. Consequently,

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}). \tag{4}$$

The curvature tensor is given by (with the first index lowered):

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^{\lambda}{}_{\nu\rho\sigma} = g_{\mu\lambda}(\partial_\rho \Gamma^{\lambda}{}_{\sigma\nu} - \partial_\sigma \Gamma^{\lambda}{}_{\rho\nu} + \Gamma^{\lambda}{}_{\rho\alpha}\Gamma^{\alpha}{}_{\sigma\nu} - \Gamma^{\lambda}{}_{\sigma\alpha}\Gamma^{\alpha}{}_{\rho\nu})$$

Since the Christoffel symbols are already linear in $h_{\mu\nu}$, the double terms may be eliminated. Again letting $g_{\mu\nu} = \eta_{\nu} + h_{\mu\nu}$ and linearizing (and switching the lower indices on the second Christoffel symbol):

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_{\rho}\Gamma^{\lambda}{}_{\nu\sigma} - \eta_{\mu\lambda}\partial_{\sigma}\Gamma^{\lambda}{}_{\nu\rho} \\ &= \frac{1}{2}(\partial_{\rho}\partial_{\nu}h_{\mu\sigma} + \partial_{\sigma}\partial_{\mu}h_{\nu\rho} - \partial_{\sigma}\partial_{\nu}h_{\mu\rho} - \partial_{\rho}\partial_{\mu}h_{\nu\sigma}) \end{aligned} \tag{6}$$

Contracting over μ and ρ gives

$$\begin{aligned} R_{\nu\sigma} &= g^{\mu\rho}R_{\mu\nu\rho\sigma} \\ &= (\eta^{\mu\rho} - h^{\mu\rho})R_{\mu\nu\rho\sigma} \\ &= \eta^{\mu\rho}R_{\mu\nu\rho\sigma} \\ &= \frac{1}{2}(\partial_{\mu}\partial_{\nu}h^{\mu}{}_{\sigma} + \partial_{\sigma}\partial_{\mu}h^{\mu}{}_{\nu} - \partial_{\sigma}\partial_{\nu}h^{\mu}{}_{\mu} - \partial_{\mu}\partial_{\mu}h_{\nu\sigma}) \end{aligned} \tag{7}$$

and renaming the lower indices,

$$R_{\mu\nu} = \frac{1}{2}(\partial_{\sigma}\partial_{\nu}h^{\sigma}{}_{\mu} + \partial_{\mu}\partial_{\sigma}h^{\sigma}{}_{\nu} - \partial_{\mu}\partial_{\nu}h - \Box h_{\mu\nu}), \tag{8}$$

where $h = h^{\mu}{}_{\mu}$ is the trace of h, and $\Box h = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$ is the D'Alembertian of flat spacetime. Finally, contracting $R_{\mu\nu}$ over the last indices produces the Ricci scalar (to first order):

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ &= (\eta^{\mu\nu} - h^{\mu\nu})R_{\mu\nu} \\ &= \frac{1}{2}(\partial_{\sigma}\partial_{\mu}h^{\sigma\mu} + \partial_{\mu}\partial_{\sigma}h^{\sigma\mu} - \partial_{\mu}\partial_{\mu}h - \Box h_{\mu\mu}) \\ &= \partial_{\mu}\partial_{\nu}h^{\mu\nu} - \Box h \end{aligned} \tag{9}$$

Our hard work finally allows us to express the Einstein tensor in terms of $h_{\mu\nu}$:

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ &= R_{\mu\nu} - \frac{1}{2}(\eta_{\mu\nu} + h_{\mu\nu}) R \\ &= \frac{1}{2}(\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\mu \partial_\nu h^{\mu\nu} + \eta_{\mu\nu} \Box h) \end{aligned} \tag{10}$$

Unfortunately, our choice of the metric $g_{\mu\nu}$ does not completely specify the coordinate system— there exists some gauge freedom. It is not the intent of this paper to delve into the gauge invariance of the metric. Rather, a gauge shall be chosen in order to make $g_{\mu\nu}$ unique (the curious reader is directed to Carroll). First off, the harmonic gauge is selected which leads to the restriction

$$g^{\mu\nu} \Gamma^\rho{}_{\mu\nu} = 0 \tag{11}$$

which, in our weak field approximation, gives

$$\begin{aligned} &\frac{1}{2} \eta^{\mu\nu} \eta^{\lambda\rho} (\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) = 0 \\ \Rightarrow \qquad &\partial_\mu h^\mu{}_\lambda - \frac{1}{2} \partial_\lambda h = 0 \end{aligned} \tag{12}$$

Substituting (12) into (10) allows the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ to be approximated to first order:

$$\Box h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \Box h = -16\pi G T_{\mu\nu}. \tag{13}$$

To simplify (13), we may define the *trace-reversed* perturbation as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \tag{14}$$

(note that $h^{\mu}{}_{\mu} = -h^{\mu}{}_{\mu}$), to which (13) becomes

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{15}$$

In the case of a vacuum, this simplifies even further to

$$\Box \bar{h}_{\mu\nu} = 0. \tag{16}$$

(16) should scream wave equation, suggesting oscillating solutions of the perturbation term are possible. Let us see if the family of plane waves satisfies (16) where, for each component $h_{\alpha\beta}$,

$$\bar{h}_{\alpha\beta} = C_{\alpha\beta} e^{ik_\sigma x^\sigma}. \tag{17}$$

$C_{\alpha\beta}$ is a constant and k^{σ} is a constant vector called the *wave vector*. For $h_{\mu\nu}$, this plane wave may be generalized to a set of plane waves:

$$\bar{h}_{\mu\nu} = C_{\mu\nu} e^{ik_\sigma x^\sigma} \tag{18}$$

where $C_{\mu\nu}$ is now a constant, symmetric (0, 2)-tensor (it is symmetric because $h_{\mu\nu}$ is symmetric). We now check that (18) is indeed a solution:

$$0 = \Box \bar{h}_{\mu\nu} = \eta^{\rho\sigma} \partial_\rho \partial_\sigma \bar{h}_{\mu\nu} = \eta^{\rho\sigma} \partial_\rho (ik_\sigma \bar{h}_{\mu\nu}) = \eta^{\sigma\rho} ik_\sigma \partial_\rho \bar{h}_{\mu\nu} = -\eta^{\rho\sigma} k_\rho k_\sigma \bar{h}_{\mu\nu} = -k_\sigma k^\sigma \bar{h}_{\mu\nu} \tag{19}$$

and since $h_{\mu\nu} \neq 0$ for the non-trivial case, so long as

$$k_\sigma k^\sigma = 0, \tag{20}$$

plane wave perturbations are acceptable. Letting $k^0 = \omega$ (this is known as the frequency of the wave), we have $k^{\sigma} = (\omega, k^1, k^2, k^3)$ and condition (20) becomes

$$\omega^2 = \delta_{ij} k^i k^j \tag{21}$$

Clearly, then, without any gauge conditions, it takes thirteen parameters to define any plane wave (ten $C_{\alpha\beta}$'s and three k^i's). However, the harmonic gauge condition requires that

$$0 = \partial_\mu \bar{h}^{\mu\nu} = \partial_\mu (C^{\mu\nu} e^{ik_\sigma x^\sigma}) = i C^{\mu\nu} k_\mu e^{ik_\sigma x^\sigma}$$

$$\Rightarrow \qquad k_\mu C^{\mu\nu} = 0 \tag{22}$$

These additional four restrictions reduce the number of independent components from ten to six. Yet this still does not specify the coordinate system. With this set of six coefficients (call them $C^{(old)}{}_{\mu\nu}$), we still have the freedom find another set $C^{(new)}{}_{\mu\nu}$ that refers to the same wave so that

$$C^{(new)}{}_{0\nu} = 0 \tag{23}$$

$$C^{(new)\mu}{}_{\mu} = 0 \tag{24}$$

Now, if we use the above convention when selecting our coefficients, this leaves only two independent coefficients of $C_{\mu\nu}$ and eliminates any gauge freedom. Consequently, selecting the harmonic gauge as well as restrictions (23) and (24) produces a unique coordinate system. To take an example, a wave traveling in the x^3 direction will have $k^\sigma = (\omega, 0, 0, \omega)$, and using our gauge restrictions, we have that the only nonzero components of $C_{\mu\nu}$ are C_{11}, C_{12}, C_{21} and C_{22}. However $C_{\mu\nu}$ is also traceless by (24) and, as stated earlier, it is symmetric. Thus, $C_{\mu\nu}$ is given by:

(25)

For obvious reasons this gauge is known as the *transverse traceless gauge*. A nice consequence of this choice is that $h_{\mu\nu}$ is traceless (since $C_{\mu\nu}$ is); thus, looking back to (14) for the relation between $h_{\mu\nu}$ and $h_{\mu\nu}$, we see that

$$\bar{h}_{\mu\nu} = h_{\mu\nu} \tag{26}$$

for the transverse traceless gauge.

Having shown that Einstein's equations predict the existence of gravitational waves wit h planar geometry, let us now examine the effect of such waves on test particles. Obviously, it is insufficient to consider a single particle as gravitational waves alter the metric and affect only the relative distances. It seems appropriate, then, to consider the relative motion of a group of free-fall particles through the geodesic deviation equation

$$\frac{D^2}{d\tau^2} = R^{\mu}{}_{\nu\rho\sigma} U^{\nu} U^{\rho} S^{\sigma} \tag{27}$$

where the four-velocities of the particles are indicated by the single vector field $U^{\mu}(x)$ (Recall the separation vector $S^{\mu} = \partial x^{\mu}/\partial s$ and τ is the proper time in the particles' inertial frame. The notion of proper time makes sense because we can assume the Einstein Equivalence Principle applies in this small region of spacetime). Assuming that our particles are moving slowly, we may take $U^{\mu}(x)$ to be a unit vector in the time direction plus powers of $h_{\mu\nu}$. But because $R^{\mu}{}_{\nu\rho\sigma}$ is already expressed to first order in $h_{\mu\nu}$, $U^{\mu}(x) = (1, 0, 0, 0)$ is a sufficient approximation. Consequently, $R^{\mu}{}_{00\sigma}$ is the only component to be calculated for (27) (or equivalently $R_{\mu 00\sigma}$):

$$R_{\mu 00\sigma} = \frac{1}{2}\partial_0\partial_0 h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{00} - \partial_\sigma\partial_0 h_{\mu 0} - \partial_\mu\partial_0 h_{\sigma 0} = \frac{1}{2}\partial_0\partial_0 h_{\mu\sigma} \tag{28}$$

where $h_{\mu 0} = 0$ (from (23)) is used in the last equality. As well, $\tau = t$ to first order and our geodesic equation simplifies to

$$\frac{\partial^2}{\partial t^2} S^{\mu} = \frac{1}{2} S^{\sigma} \frac{\partial^2}{\partial t^2} h^{\mu}{}_{\sigma} \tag{29}$$

For our wave traveling in the x^3 direction, (29) informs us that only S^1 and S^2 will be perturbed, meaning only those particles in the plane perpendicular to the propagating wave will be affected (since our system is a perturbed flat space, it make sense to speak of such geometry). For this wave let us and define $C_+ \equiv C_{11}$ and $C_\times \equiv C_{12}$. Considering C_+ alone (setting $C_\times = 0$), (29) gives the two uncoupled equations

$$\frac{\partial^2}{\partial t^2} S^1 = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2} (C_+ e^{ik_\sigma x^\sigma}) \tag{30}$$

$$\frac{\partial^2}{\partial t^2} S^2 = -\frac{1}{2} S^2 \frac{\partial^2}{\partial t^2} (C_+ e^{ik_\sigma x^\sigma}) \tag{31}$$

whose solutions, to first order, are:

$$S^2 = (1 - \frac{1}{2} C_+ e^{ik_\sigma x^\sigma}) S^2(0) \tag{32}$$

$$S^1 = (1 + \frac{1}{2} C_+ e^{ik_\sigma x^\sigma}) S^1(0) \tag{33}$$

It is clear that the distance between particles in the x^1 direction will vary sinusoidally, with a similar motion in the x^2 direction. This phenomenon is perhaps best illustrated by a group of particles in the x^1-x^2 plane initially in a circle. As the gravitational wave passes through, the circle squashes vertically and horizontally (fig. 1). Similar solutions occur when $C_+ = 0$ instead:

$$S^1 = S^1(0) + \frac{1}{2} C_\times e^{ik_\sigma x^\sigma} S^2(0) \tag{34}$$

$$S^2 = S^2(0) + \frac{1}{2} C_\times e^{ik_\sigma x^\sigma} S^1(0) \tag{35}$$

Fig. 1. *(Top) Influence of C_+ mode on ring of particles.*
Fig. 2. *(Bottom) Influence of $C_\times$ mode on ring of particles.*

Here, the ring squashes diagonally (fig. 2) (the subscripts on C should now make sense). Since any wave traveling in the x^3 direction can be characterized by the independent coefficients C_+ and $C_\times$, these numbers considered separately determine the linear polarized normal modes of the gravitational wave. Note that these modes are at 45° to each other, rather than 90° as is the case for electromagnetic waves.

INTERFEROMETRIC GRAVITATIONAL WAVE DETECTORS

Clearly, gravitational waves alter relative distances through local perturbations in the spacetime metric. It stands to reason, then, that any device created to measure these waves must be capable of detecting such changes. In fact, all modern detectors are based upon the design of an interferometer created by Michelson and Morely over one hundred years ago (fig. 3). The design is simple: laser light is split into two perpendicular paths by a beam splitter. A mirror placed at the end of each path reflects the light back to the splitter, where it is recombined, and its intensity measured with a photodiode. If the mirrors are kept equidistant from the splitter, the two wave fronts return at the same time and constructively interfere when they recombine. This results in maximum output intensity. Similarly, if the mirrors are moved so that their relative distance to the splitter remains constant, the wave fronts will still recombine "in synch." If, however, the relative distance is changed, the waves will return out of step, some destructive interference will occur and the measured intensity will drop. The dependence of power output to arm length difference is graphed in fig. 4. It is stated without proof that the relation between output power and arm length is

Fig. 3. *Schematic diagram of Michelson-Morely interferometer.*

$$P_{out} = P \cos^2 \frac{2\pi\Delta L}{\lambda} \tag{36}$$

where P_{in} is the input power of the laser, ΔL is the difference in length between the arms and λ is the wavelength of laser light (this result can be found in any optics text). An alternative formulation of (36) involves the phase difference $\Delta\phi$ between the two wavefronts: $P_{out} = P_{in}cos^2(\Delta\phi)$. Thus, the Michelson-Morely interferometer is able to translate changes in distance, within fractions of the laser's wavelength, to changes in light intensity.

Interferometric gravitational wave detectors work on the same principle. In this case, the mirrors are attached to large masses (usually 30-100 kg) hung as pendulums at the end of each arm. Gravitational waves passing through the plane of the detector will then cause the masses to move relative to each other. A phase shift between the two wave fronts will occur and a change in intensity will be recorded at the photodiode.

Fig. 4. *Output power vs. arm length difference from a Michelson interferometer*

Let us quantify this process using a specific case. Suppose there exists a plane gravitational wave traveling perpendicular to the plane of the detector. Without loss of generality, we may orient our coordinate system so that the arms of the detector are aligned with the x^1 and x^2 axes, the beam splitter is positioned at the origin, and the wave propagates parallel to the x^3 axis. We will also assume that the arms are at equal distances to the splitter at $x^0 = 0$. As noted earlier, such a plane wave is completely characterized by the coefficients of the normal modes, C_+ and $C_\times$. However, looking back to (34) and (35), it is clear that the $C_\times$ mode alters the x^1 and x^2 distances equally (provided distances are equal at t = 0), resulting in zero phase shift between the two beams. We can thus neglect this component of the wave, making h_{11} and h_{22} the only nonzero components of the perturbation metric.

Consider light traveling to and from the beam splitter in the arm along the x^1 axis. The interval between two spacetime events linked by a light beam is given as

$$\begin{aligned} ds^2 = 0 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \\ &= -cdt^2 + (1 + C_+ e^{ik_\sigma x^\sigma})(dx^1)^2 \end{aligned} \tag{37}$$

where $h_{11} = C_+ exp(ik_\sigma x^\sigma)$ as before. Note that we have reintroduced c as the speed of light so that this physical calculation is more meaningful. The travel time of the light in the x^1 arm from the splitter to the mirror can be found by integrating the square root of (37)

$$\int_0^{\tau_{out}} dt = \frac{1}{c}\int_0^L \sqrt{1 + C_+ e^{ik_\sigma x^\sigma}}\, dx^1 \approx \frac{1}{c}\int_0^L (1 + \frac{1}{2}C_+ e^{ik_\sigma x^\sigma})\, dx^1 \tag{38}$$

Here we have truncated the binomial expansion of the square root past first order since $|h| \ll 1$. Recall that $k^\sigma = (\omega, 0, 0, \omega)$, making $exp(ik_\sigma x^\sigma) = exp[i\omega(x^0 + x^3)]$; however $x^3 = 0$ in the plane of the detector. Thus,

$$\begin{aligned} \frac{1}{c}\int_0^L (1 + \frac{1}{2}C_+ e^{ik_\sigma x^\sigma})\, dx^1 &= \frac{1}{c}\int_0^L (1 + \frac{1}{2}C_+ e^{i\omega x^0})\, dx^1 \\ &= \frac{1}{c}\int_0^L (1 + \frac{1}{2}C_+ e^{i\omega \frac{x^1}{c}})\, dx^1 \end{aligned} \tag{39}$$

where the substitution $t = x^1/c$ was made in the final line. The integral is now easily solved:

$$\int_0^{\tau_{out}} dt = \frac{L}{c} + \frac{C_+}{2i\omega}[e^{i\omega \frac{L}{c}} - 1] \tag{40}$$

In a similar fashion, the equation for the return trip can be written as (using the substitution $t = (2L - x)/c$)

$$\int_{\tau_{out}}^{\tau_{rt}} dt = -\frac{1}{c}\int_L^0 (1 + \frac{1}{2}C_+ e^{ik_\sigma x^\sigma})$$

$$= \frac{L}{c} + \frac{C_+}{2i\omega} e^{2i\omega\frac{L}{c}} [1 - e^{-i\omega\frac{L}{c}}] \tag{41}$$

with τ_{rt} being the time of the round trip. After a similar analysis for the arm in the x^2 direction, we find that the difference in travel times between the two arms is

$$\Delta\tau = C_+ \tau_{rt0} e^{\frac{i\omega\tau_{rt0}}{2}} sinc(\omega\tau_{rt0}/2) \tag{42}$$

with $\tau_{rt0} \equiv 2L/c$ and the function $sinc\ x \equiv (1/\pi x)sin(\pi x)$. The phase shift can be expressed by multiplying this time difference by the angular frequency of the light, $2\pi c/\lambda$

$$\Delta\phi = C_+ \tau_{rt0} \frac{2\pi c}{\lambda} sinc(\omega\tau_{rt0}/2\pi) e^{i\frac{\omega}{2}\tau_{rt0}} \tag{43}$$

This will register as a drop in intensity of $cos^2(\Delta\phi)$.

Working backwards, then, this detector allows the amplitude of a gravitational wave to be easily quantified by a change in light output (at least in principle).

GRAVITATIONAL WAVE SOURCES

Now that it is understood how gravitational waves propagate in a vacuum, it remains to examine how these waves are generated in the first place. Here, we cannot assume there exists a vacuum (to the contrary, matter must be present to generate waves); thus, $T_{\mu\nu} \neq 0$ and our linearized Einstein equations take the form of (15):

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{44}$$

Without getting bogged down in PDEs, the general solution to (44) is, in integral form,

$$\bar{h}_{\mu\nu}(t,x) = 4G\int \frac{1}{|x-y|} T_{\mu\nu}(t-|x-y|, y)\, d^3y \tag{45}$$

where $x^0 = t$. Note that the integration is only over the spacelike coordinates and, similar to electromagnetic fields, $T_{\mu\nu}$ is evaluated at the retarded time $t_r = t - |x - y|$. Our goal is to solve (45) explicitly for the case of a distant, isolated, non-relativistic source. We start by taking the Fourier transform, with respect to time only, of $h_{\mu\nu}(x,t)$:

$$\begin{aligned} \bar{h}_{\mu\nu}(\omega,x) &= \frac{1}{\sqrt{2\pi}}\int dt\, e^{i\omega t}\bar{h}_{\mu\nu}(t,x) \\ &= \frac{4G}{\sqrt{2\pi}}\int dt\, d^3y\, e^{i\omega t}\frac{T_{\mu\nu}(t-|x-y|,y)}{|x-y|} \\ &= \frac{4G}{\sqrt{2\pi}}\int dt_r\, d^3y\, e^{i\omega t_r} e^{i\omega|x-y|}\frac{T_{\mu\nu}(t_r,y)}{|x-y|} \end{aligned} \tag{46}$$

where the last line is a change of variables from t to t_r. With this, note that

$$T_{\mu\nu}(\omega,y) = \frac{1}{\sqrt{2\pi}}\int dt_r\, e^{i\omega t_r} T_{\mu\nu}(t_r,y) \tag{47}$$

and hence,

$$\bar{h}_{\mu\nu}(\omega,x) = 4G\int d^3y\, e^{i\omega|x-y|}\frac{T_{\mu\nu}(\omega,y)}{|x-y|}. \tag{48}$$

Our approximation that the source is far away implies that from our (spatial) vantage point of x, the object is small compared to its distance R. Similarly, we will also assume that the object is moving slowly compared to the speed at which the radiation is emitted. This way, $e^{i\omega|x-y|}/|x-y|$ can be approximated as $e^{i\omega R}/R$ when integrating over the volume of the source, giving

$$\bar{h}_{\mu\nu}(\omega,x) = 4G\frac{e^{i\omega R}}{R}\int d^3y\, T_{\mu\nu}(\omega,y). \tag{49}$$

Furthermore, the harmonic gauge condition (11) translates in Fourier space to

$$\bar{h}^{0\nu} = \frac{i}{\omega}\partial_j \bar{h}^{j\nu} \tag{50}$$

where j = 1, 2, 3 (this was found by applying the Fourier transformation to (11)). We may therefore limit our attention to the spacelike components of $h_{\mu\nu}(\omega,x)$. Doing so for (49), we manipulate the right side by integrating by parts in reverse:

$$\int d^3y\, T^{ij}(\omega,\ y) = \int \partial_k(y^i T^{kj})\, d^3y - \int y^i(\partial_k T^{kj})\, d^3y. \tag{51}$$

The first term on the right is a surface integral and is zero since the source is isolated. We can relate the second term to T^{0j} through the Fourier space version of $\partial_\mu T^{\mu\nu} = 0$ (conservation of $T^{\mu\nu}$),

$$-\partial_k T^{ku} = i\omega T^{0\mu}, \tag{52}$$

which gives us

$$\int d^3y\, T^{ij}(\omega,y) = i\omega\int y^i T^{0j}\, d^3y = \frac{i\omega}{2}\int (y^i T^{0j} + y^j T^{0i})\, d^3y \tag{53}$$

where the last equality comes from the symmetry of T^{ij}. Employing integration by parts in reverse to this result and then using the conservation of $T^{\mu\nu}$ leads to the following:

$$\int d^3y\, T^{ij}(\omega,y) = \frac{i\omega}{2}\int [\partial_l(y^i y^j T^{0l}) - y^i y^j(\partial_l T^{0l})]\, d^3y = -\frac{\omega^2}{2}\int y^u y^j T^{00}\, d^3y. \tag{54}$$

Finally, by defining the quadrupole moment tensor of the source as

$$q_{ij}(t) \equiv 3\int y^i y^j T^{00}(t,y)\, d^3y \tag{55}$$

$h_{ij}(\omega,x)$ becomes

$$\bar{h}_{ij}(\omega,x) = -\frac{2G\omega^2}{3}\frac{e^{i\omega R}}{R}q_{ij}(\omega) \tag{56}$$

or, using the inverse Fourier transform,

$$h_{ij}(t,x) = \bar{h}_{ij}(t,x) = \frac{2G}{3R}\frac{d^2}{dt^2}q_{ij}(t_r) \tag{57}$$

Thus, the gravitational waves produced by a distant, isolated, nonrelativistic object are proportional in amplitude to the second time derivative of the quadrupole moment tensor evaluated at the retarded time $t_r = t - R$.

But what type of sources emit gravitational radiation? A typical example is a binary neutron star system (i.e. two neutron stars that orbit each other) as seen in fig. 5. Assume such a system exists in the Virgo cluster (a hot spot for neutron stars) 4.5×10^{23} m away. Here, each star has a mass 3×10^{30} kg and orbits in a circular fashion with a distance of 20 km to the centre of mass of the system. Circular motion implies a constant orbital frequency which we will take to be 800 s^{-1} (these are standard numbers for such a system). Orient the coordinate system so that the normal to the orbital plane is in the x^3 direction. Then, using (55) the components of the quadrupole moment are easily calculated to be

Fig. 5. *Schematic of a binary star system.*

$$q_{22} = A\sin^2(800t)$$
$$q_{11} = A\cos^2(800t)$$
$$q_{12} = q_{21} = A\cos(800t)\sin(800t)$$

where $A \approx 10^{33}$. The rest of the components are zero. Finally, the components of h_{ij} can be found from (57) to be

$$-h_{11} = h_{22} = B cos(1600t_r)$$
$$h_{12} = h_{21} = B\sin(1600t_r)$$

Here, B $\approx 10^{-21}$. Needless to say, with $|h_{\mu\nu}|$ on the order of 10^{-21}, the effects of such a wave are astonishingly small, yet these binary sources are predicted to produce some of the strongest waves experienced from Earth. Even orbiting black holes (a system similar to the above, only with two black holes in their place) are estimated to perturb spacetime at most one part in 10^{20}!

It is questionable whether even the most sensitive interferometric detector could recognize these small perturbations. Provided that appropriate measures are taken to reduce noise levels, however, most specialists feel such sensitivities are in fact possible (see Blair and Saulson). The following section looks at two of these noise sources and how their influence can be reduced.

NOISE REDUCTION

Photon Shot Noise

Recall from our previous discussion that gravitational waves are detected through relative changes in the output intensity of laser light from a Michelson-Morely interferometer. Thus, an equivalent question to “How small can be detected?” is “How small of a change in optical power can be detected?” Based on the particle nature of light, we see that there is indeed a lower bound to this precision, the result of a phenomenon known as photon shot noise. To explain, first note that output power is proportional to the number of photons N arriving at the photodiode per unit time. Because the power output of the laser is fixed, we do expect a mean number of photons N per interval; still, the arrival of each photon is random and independent of other photons. Such a process of discreet, independent events is characterized by a Poisson distribution function

$$p(N) = \frac{\overline{N}^{N} e^{-\overline{N}}}{N!}. \tag{58}$$

When N »1, this distribution can be approximated by a Gaussian distribution with standard deviation $\sigma = N$. Thus by measuring the number of photons per time interval, we would discover a fractional fluctuation in these measurements of

$$\frac{\sigma_{\overline{N}}}{\overline{N}} = \frac{\sqrt{\overline{N}}}{\overline{N}} = \frac{1}{\sqrt{\overline{N}}}. \tag{59}$$

With this result in hand, we can now proceed to calculate fluctuations in output power. Each photon carries an energy of $\hbar\omega = 2\pi\hbar c/\lambda$. The photon flux can therefore be written as

$$\overline{N} = \frac{\lambda}{2\pi\hbar c} P_{out} \tag{60}$$

In order to determine the size of expected power fluctuations, we must select the mean output power of the interferometer (i.e. adjust the arms to the desired point on fig. 4). Logically, the best position would be $P_{in}/2$, as the sensitivity dP_{out}/dL to changes in arm length is at a maximum with

$$\frac{dP_{out}}{dL} = \frac{2\pi}{\lambda} P. \tag{61}$$

At this operating point, the mean number of photons per measurement is $N = (\lambda/4\pi\hbar c)P_{in}$ and the fractional photon fluctuation will be given by

$$\frac{\sigma_{\overline{N}}}{\overline{N}} = \sqrt{\frac{4\pi\hbar c}{\lambda P}} \tag{62}$$

However, such power fluctuations will be interpreted as relative movement δL in the arms of the detector. This "false motion" is given by the fractional photon fluctuation divided by the fractional output power change per unit position difference:

$$\sigma_{\delta L} = \frac{\sigma \overline{N}}{\overline{N}} / \frac{1}{P_{out}} \frac{dP_{out}}{dL} = \sqrt{\frac{\hbar c \lambda}{4\pi P}} \tag{63}$$

Finally, if h evoked a fractional change in length of δL, we have that $\delta L = Lh$. So these fluctuations in brightness can be seen equivalently as noise in h,

$$\sigma_h = \frac{1}{L}\sqrt{\frac{\hbar c \lambda}{4\pi P}}. \tag{64}$$

(64) can be expressed with some typical numbers (see Saulson):

$$\sigma_h = 3.7 \times 10^{-22} \frac{600\,km}{L} \sqrt{\frac{\lambda}{0.545\,\mu m}} \sqrt{\frac{1W}{P}}. \tag{65}$$

600 km is a standard optical arm length for most interferometers designed to date (using a clever technique to be discussed later), while 0.545 μm is the wavelength of the most commonly used laser, the green argon ion laser. It is apparent that photon shot noise presents a serious threat to waves that have magnitudes of 10^{-21} at best. The only chance in the near future for reducing this noise lies in increasing the power output of the argon laser. While present argon lasers have a power output of about 1 Watt, new versions are being designed with P_{in} ~ 1000 Watts which could reduce σ_h by

another order of magnitude.

Brownian Motion

The effects of Brownian motion (the jostled movement of matter in response to molecular collisions) are usually considered only for microscopic particles. However, the precision required by these detectors forces us to investigate the interactions of the test masses with the surrounding gas molecules. Suppose, for example, the detector contains suspended rectangular plate masses with cross-sectional area A (see fig. 6). The pressure of the gas surrounding the masses is given by $p = nk_BT$ (where n is the number density of molecules, k_B is Boltzmann's constant, and T is the absolute temperature). Consider the case when p is so low that the mean free path for the gas molecules is large compared to the dimensions of the test mass. Here, we can neglect intermolecular collisions and concentrate on molecule-plate interactions.

Fig. 6. *Schematic of test mass suspended in gas.*

When at rest, the plate will receive an equal number of collisions from either side, on average. The average rate of collisions on one side will be given by

$$\overline{N} = \frac{1}{4} n\bar{v}A \qquad (66)$$

where v is the average velocity of a gas molecule (see any introductory statistical mechanics book for this derivation). This collision rate involves counting discreet, independent events per unit time; thus, any fluctuations will obey Poisson statistics. As for shot noise, the fractional fluctuation of collisions per unit time will be

$$\frac{\sigma_{\overline{N}}}{\overline{N}} = \frac{1}{\sqrt{\overline{N}}}. \tag{67}$$

Also note that if each molecule reflects elastically off the surface, there will be a mean force exerted on one side equal to pA, or equivalently,

$$\overline{F}_{+} = nk_B TA. \tag{68}$$

Since the force exerted should be proportional to the collision rate, we expect the fractional fluctuation in force to be equal to the fractional fluctuation of collisions:

$$\frac{\sigma_{\overline{F}}}{\overline{F}} = \frac{\sigma_{\overline{N}}}{\overline{N}} = \frac{1}{\sqrt{\overline{N}}} \tag{69}$$

so that the force noise is

$$\begin{aligned} \sigma_{\overline{F}} &= \frac{\overline{F}}{\sqrt{\overline{N}}} \\ &= 2k_B T\sqrt{\frac{nA}{\overline{v}}} \end{aligned} \tag{70}$$

Naturally, any variation in force will result in changes to the test mass position. But (70) demonstrates that the force noise can be lowered by reducing T, n or A and in fact, most detector designs minimize all three. The chamber containing the mass is invariably super-cooled so that T is near zero, while the pressure inside is reduced to one trillionth of an atmosphere. Similarly, spherical test masses are used instead of plates so that the surface area to mass ratio is minimized.

It should be stated that the above discussion is only an emaciated survey of noise sources and their reduction. Other noise sources include: power output and wavelength fluctuations in the laser, seismic noise, tidal effects of the sun and moon, and the internal vibrational modes of the test masses. Entire volumes have been written on the subject. The interested reader is directed to Saulson and/or Blair.

LIGO

Fig. 7. *Schematic of LIGO detector.*

The next few years should be quite exciting for gravitational wave detection. Within that time, the first three detectors will become operational: two in the United States and one in Italy. The LIGO project (Laser Interferometer Gravitational-wave Observatory), lead by a Caltech/MIT consortium, involves detectors currently being completed in Hanford, Washington and Livingston, Louisiana. Their (overly) simplified design (see fig. 8) is similar to the setup of fig. 3—each arm, which spans 4 km, is kept under a high vacuum (1×10^{-12} atm) to reduce the Brownian noise described above. Note that 4 km is well below an arm length of 600 km mentioned earlier. To achieve this length, the designers cleverly added an extra mirror to the beginning of each arm, effectively creating a Fabry-Perot cavity. Light bounces back and forth an average of 150 times before exiting the interferometer, changing a physical length of 4 km to an optical path length of 600 km. The building of two detectors is yet another noise elimination technique. It is predicted that high-amplitude noise will be able to find its way into the detector on occasion and two detectors will allow any signals to be double-checked. A true gravitational wave front would pass through both detectors nearly simultaneously, with both systems registering the disturbance. The chance of noise occurring at the same time in both locations, however, is very slim. System tests at both the Hanford and Livingston detectors have been ongoing for the past two years, with Hanford due to go online in May 2002 and Livingston later in 2003.

It is hoped that these two detectors, along with one in Italy (VIRGO), will not only be able to show the existence of gravitational waves, but will also pinpoint such sources in the sky (this can be done by measuring the time difference between each detector first receiving a signal, and then using triangulation). With this information, astronomers can then direct their attention to these regions for further study. Many speak of gravitational wave detectors as providing a turning point in our understanding of the universe. When one considers that electromagnetic radiation only yields about 10% of an astronomical object's information, it is clear how potentially biased our knowledge is. If successful, gravitational wave detectors might well take us to new frontiers in science.

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