On the nonoscillatory phase function for Legendre's differential equation

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Abstract

We express a certain complex-valued solution of Legendre's differential equation as the product of an oscillatory exponential function and an integral involving only nonoscillatory elementary functions. By calculating the logarithmic derivative of this solution, we show that Legendre's differential equation admits a nonoscillatory phase function. Moreover, we derive from our expression an asymptotic expansion useful for evaluating Legendre functions of the first and second kinds of large orders, as well as the derivative of the nonoscillatory phase function. Our asymptotic expansion is not as efficient as the well-known uniform asymptotic expansion of Olver; however, unlike Olver's expansion, it coefficients can be easily obtained. Numerical experiments demonstrating the properties of our asymptotic expansion are presented.

Keywords: special functions, fast algorithms, nonoscillatory phase functions

1. Introduction

The Legendre functions of degree $\nu \in \mathbb{C}$ — that is, the solutions of the second order linear ordinary differential equation

$$y''(z) - \frac{2z}{1-z^2}y'(z) + \frac{\nu(\nu+1)}{1-z^2}y(z) = 0 \tag{1}$$

— appear in numerous contexts in physics and applied mathematics. For instance, they arise when certain partial differential equations are solved via separation of variables, they are often used to represent smooth functions defined on bounded intervals, and their roots are the nodes of Gauss-Legendre quadrature rules. For our purposes, it is convenient to work with the functions $\bar{P}_\nu$ and $\bar{Q}_\nu$ defined for $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\nu \geq 0$ via the formulas

$$\bar{P}_\nu(\theta) = \sqrt{\left(\nu + \frac{1}{2}\right)}\, P_\nu(\cos(\theta))\sqrt{\sin(\theta)} \tag{2}$$

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and

$$\bar{Q}_\nu(\theta) = -\frac{2}{\pi}\sqrt{\left(\nu+\frac{1}{2}\right)}\; Q_\nu(\cos(\theta))\sqrt{\sin(\theta)}, \tag{3}$$

where P_ν and Q_ν are the Legendre functions of the first and second kinds of degree ν, respectively. The functions (2) and (3) are solutions of the second order linear ordinary differential equation

$$y''(\theta) + \left(\left(\nu+\frac{1}{2}\right)^2 + \frac{1}{4}\operatorname{cosec}^2(\theta)\right) y(\theta) = 0 \quad \text{for all} \quad 0 < \theta < \frac{\pi}{2}. \tag{4}$$

By a slight abuse of terminology, we will refer to (4) as Legendre's differential equation.

The coefficient of y in (4) is positive and increases with ν, with the consequence that $\bar{P}_\nu$ and $\bar{Q}_\nu$ are highly oscillatory when ν is of large magnitude. It has long been known that despite this there exist phase functions for (4) which are nonoscillatory in some sense. In particular, there is a nonoscillatory function α_ν whose derivative is positive on $\left(0,\frac{\pi}{2}\right)$ and such that

$$\bar{P}_\nu(\theta) = \sqrt{W}\,\frac{\cos(\alpha_\nu(\theta))}{\sqrt{\alpha_\nu'(\theta))}} \tag{5}$$

and

$$\bar{Q}_\nu(\theta) = \sqrt{W}\,\frac{\sin(\alpha_\nu(\theta))}{\sqrt{\alpha_\nu'(\theta)}}, \tag{6}$$

where W is the Wronskian

$$W = \frac{2}{\pi}\left(\nu+\frac{1}{2}\right) \tag{7}$$

of the pair $\bar{P}_\nu$, $\bar{Q}_\nu$. By differentiating the expressions

$$\frac{\bar{Q}_\nu(\theta)}{\bar{P}_\nu(\theta)} = \tan\left(\alpha_\nu(\theta)\right) \quad \text{and} \quad \frac{\bar{P}_\nu(\theta)}{\bar{Q}_\nu(\theta)} = \operatorname{cotan}\left(\alpha_\nu(\theta)\right), \tag{8}$$

at least one of which is sensible at any point in $\left(0,\frac{\pi}{2}\right)$ since $\bar{P}_\nu$ and $\bar{Q}_\nu$ cannot vanish simultaneously there, we obtain

$$\alpha_\nu'(\theta) = \frac{W}{\left(\bar{P}_\nu(\theta)\right)^2 + \left(\bar{Q}_\nu(\theta)\right)^2}. \tag{9}$$

That (9) is nonoscillatory is well known. Indeed, this can be seen in a straightforward fashion from Olver's uniform asymptotic expansions

$$\bar{P}_\nu(\theta) \sim \sqrt{\lambda\theta}\left(J_0\left(\lambda\theta\right)\sum_{j=0}^{\infty}\frac{A_j(-\theta^2)}{\lambda^{2j}} - \frac{\theta}{\lambda}J_1\left(\lambda\theta\right)\sum_{j=0}^{\infty}\frac{B_j(-\theta^2)}{\lambda^{2j}}\right) \quad \text{as} \quad \nu\to\infty \tag{10}$$

and

$$\bar{Q}_\nu(\theta) \sim \sqrt{\lambda\theta}\left(Y_0\left(\lambda\theta\right)\sum_{j=0}^{\infty}\frac{A_j(-\theta^2)}{\lambda^{2j}} - \frac{\theta}{\lambda}Y_1\left(\lambda\theta\right)\sum_{j=0}^{\infty}\frac{B_j(-\theta^2)}{\lambda^{2j}}\right) \quad \text{as} \quad \nu\to\infty \tag{11}$$

for the Legendre functions (a derivation of these expansions can be found in Chapter 5 of [13]). In (10) and (11), $\lambda = \nu + \frac{1}{2}$, $A_0(\xi) = 1$, and the remaining coefficients $A_1, A_2, \ldots$ and $B_0, B_1, \ldots$

are defined via the formulas

$$B_k(\xi) = -A_k'(\xi) + \frac{1}{|\xi|^2}\int_\xi^0 \left(\frac{1}{16}\left(\operatorname{cosec}^2\left(\sqrt{|\tau|}\right) + \frac{1}{\tau}\right)\ A_k(\tau)\ - \frac{A_k'(\tau)}{2\sqrt{|\tau|}}\right)\ d\tau \tag{12}$$

and

$$A_{k+1}(\xi) = -\xi B_k'(\xi) - \frac{1}{16}\int_\xi^0 \left(\operatorname{cosec}^2\left(\sqrt{|\tau|}\right) + \frac{1}{\tau}\right)\ B_k(\tau)\ d\tau. \tag{13}$$

By plugging (10) and (11) into (9) and taking (7) into account, we obtain

$$\alpha_\nu'(\theta) = \frac{2}{\pi\theta}\frac{1}{\left(J_0(\lambda\theta)\right)^2 + \left(Y_0(\lambda\theta)\right)^2} + \mathcal{O}\left(\frac{1}{\nu}\right) \quad \text{as} \quad \nu \to \infty. \tag{14}$$

The function

$$\frac{2}{\pi t}\frac{1}{\left(J_\mu(t)\right)^2 + \left(Y_\mu(t)\right)^2} \tag{15}$$

is the derivative of a phase function for the normal form

$$y''(z) + \left(1 - \frac{\frac{1}{4} - \mu^2}{z^2}\right) y(z) = 0 \tag{16}$$

of Bessel's differential equation (see, for example, [3]). It is well known that its reciprocal is completely monotonic (see Section 2.1 for a brief discussion of completely monotonic functions). This follows, for instance, from Nicholson's integral formula

$$\left(J_\mu(z)\right)^2 + \left(Y_\mu(z)\right)^2 = \frac{8}{\pi^2}\int_0^\infty K_0\left(2z\sinh(t)\right)\cosh(\mu t)\ dt, \tag{17}$$

a derivation of which can be found in Section of Chapter XIII of [15]. This is typically what is meant when it is said that Bessel's equation admits a nonoscillatory phase function. Figure 1 displays plots of the function (15) for two values of μ.

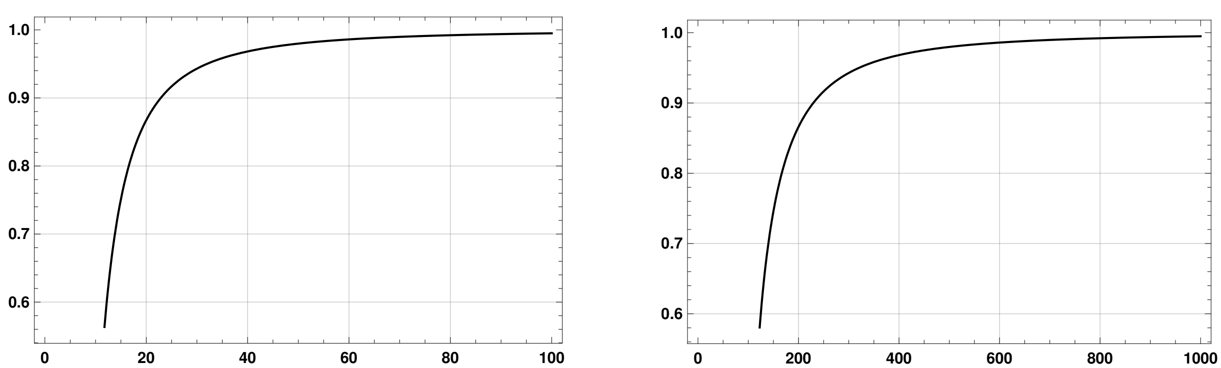

Figure 1: On the left is a plot of the function (15) on the interval $(0, 100)$ when $\mu = 10$. On the right is a plot of the same function on the interval $(0, 1000)$ when $\mu = 100$. The reciprocals of these functions are completely monotonic.

In the case of Legendre's differential equation, (14) strongly suggests that α_ν' itself is completely monotonic. This appears to be the case, although the authors are not aware of a proof to this effect in the literature and will not give one here (here, we adopt a different notion of nonoscillatory which is more relevant in numerical calculations). Figure 2 contains plots of α_ν' for two different values of ν. No doubt, many other second order differential equations defining special functions of interest have the property that they admit a phase function whose derivative is either completely monotonic or the reciprocal of a completely monotone function. However, this and other similar notions of nonoscillatory are unsatisfactory for at least two reasons. First, there are many differential equations which admit phase functions which are emphatically nonoscillatory, but whose derivatives are neither completely monotonic nor the reciprocal of a completely monotonic function. An example is given by

$$y''(t) + \frac{1+\gamma^2}{(1-t^2)^2} y(t) = 0 \quad \text{for all} \quad -1 < t < 1, \tag{18}$$

which admits a phase function α_0 such that

$$\alpha_0'(t) = \frac{\gamma}{1-t^2}. \tag{19}$$

Second, from the point of view of numerical analysis, the complete monotonicity of phase functions representing solutions of second order differential equations is of little interest. Of much more fundamental importance are estimates of the complexity of representing and evaluating the phase functions.

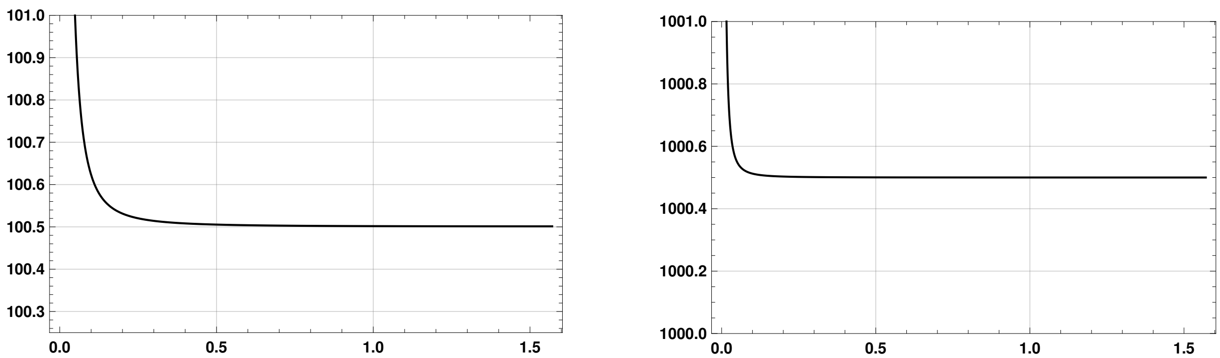

Figure 2: On the left is a plot of the function α_ν' defined via (9) when $\nu = 100$, and on the right is a plot of α_ν' when $\nu = 1000$. We conjecture that $\alpha_\nu(t)'$ is completely monotonic as a function of t whenever $\nu \geq 0$.

In [12] and [4], a notion of "nonoscillatory phase function" for second order differential equations of the form

$$y''(x) + \gamma^2 q(x) y(x) = 0 \quad \text{for all} \quad a < x < b \tag{20}$$

which is much more relevant in the context of numerical calculations is considered. In those articles, it is shown that under mild conditions on q, there exist a positive constant μ, a function α_∞ which is nonoscillatory, and a basis $\{u_\infty, v_\infty\}$ in the space of solutions of the differential equation (20) such that

$$u_\infty(x) = \frac{\cos(\alpha_\infty(x))}{\sqrt{\alpha_\infty'(x)}} + O\left(\exp(-\mu\gamma)\right) \tag{21}$$

and

$$v_\infty(x) = \frac{\sin(\alpha_\infty(x))}{\sqrt{\alpha_\infty'(x)}} + O\left(\exp(-\mu\gamma)\right). \tag{22}$$

The function α_∞ is nonoscillatory in the sense that it can be represented using a series expansion the number of terms in which depends on the complexity of q but not on the parameter γ. In other words, α_∞ represents the solutions of (20) with $\mathcal{O}\left(\exp(-\mu\gamma)\right)$ accuracy using $\mathcal{O}(1)$-term expansions. Moreover, a reliable and efficient numerical algorithm for the computation of α_∞ which runs in time independent of γ and only requires knowledge of the values of q on the interval (a, b) is introduced in [2]. Much like standard WKB estimates, the results of [12, 4, 2] easily apply to large class of differential equations whose coefficients are allowed to vary with γ — that is, equations of the more general form

$$y''(x) + q(x, \gamma)y(x) = 0 \tag{23}$$

— assuming that q satisfy certain innocuous conditions independent of γ.

The framework of [12, 4, 2] applies to (1) and it can be used to, among other things, evaluate Legendre functions of large orders and their zeros in time independent of degree. However, in the case of Legendre's differential equation a result along these lines which is not asymptotic in nature (unlike formulas (21) and (22)) can be obtained. To that end, after dispensing with certain preliminaries in Section 2 of this article, we factor the solution φ_ν of (4) defined via

$$\varphi_\nu(\theta) = \bar{P}_\nu(\theta) + i\bar{Q}_\nu(\theta) \tag{24}$$

as

$$\varphi_\nu(\theta) = -i\sqrt{\lambda \sin(\theta)} \exp(i(\nu + 1)\theta)F_\nu(\theta), \tag{25}$$

where F_ν is defined via an integral formula involving only nonoscillatory elementary functions. This is done in Section 3. Since α'_ν is related to F_ν through the formula

$$\alpha'_\nu(\theta) = \frac{\pi}{2\sin(\theta)} |F_\nu(\theta)|^{-2}, \tag{26}$$

which can be readily seen from (7), (9), (24) and (25), this provides a rather strong statement to the effect that α'_ν is nonoscillatory. In Section 4, we use our integral formula to derive an asymptotic formula for φ_ν. Our expansion is not as efficient as Olver's uniform asymptotic expansions for Legendre functions, but it has the advantage that its coefficients can be readily computed. Numerical experiments demonstrating the properties of these expansions are discussed in Section 5. We conclude this article with brief remarks in Section 6.

2. Preliminaries

2.1. Completely monotonic functions

We say that a smooth function $f : (0, \infty) \to \mathbb{R}$ is completely monotonic if

$$(-1)^n f^{(n)}(x) \geq 0 \tag{27}$$

for all nonnegative integers n and all $x > 0$. Obviously, the function $f(t) = t$ is completely monotonic. A theorem of Bernstein (a proof of which can be found in Chapter IV of [16], among many other sources) asserts that f is completely monotonic if and only if there exists a nonnegative Borel measure μ on $(0, \infty)$ such that

$$f(t) = \int_0^\infty \exp(-tx)\ d\mu(x) \tag{28}$$

where the integral converges for all $0 < x < \infty$. It follows immediately from this that the product of two completely monotonic functions is completely monotonic.

2.2. The Lipschitz-Hankel integrals

The formulas

$$P_\nu(\cos(\theta)) = \frac{1}{\Gamma(\nu+1)} \int_0^\infty \exp(-\cos(\theta)x) J_0(\sin(\theta)x) x^\nu \, dx \tag{29}$$

and

$$-\frac{2}{\pi} Q_\nu(\cos(\theta)) = \frac{1}{\Gamma(\nu+1)} \int_0^\infty \exp(-\cos(\theta)x) Y_0(\sin(\theta)x) x^\nu \, dx \tag{30}$$

hold when $0 < \theta < \pi/2$ and $\mathrm{Re}(\nu) > -1$. Here, of course, Γ denotes the Gamma function. Derivations of (29) and (30) can be found in Chapter 13 of [15]; they can also be found as Formulas 6.628(1) and 6.628(2) in [10].

2.3. The Laplace transforms of $\left(x^2 + bx\right)^{-1/2}$ and x^ν

For complex numbers z and b such that $\mathrm{Re}(z) > 0$ and $|\arg(b)| < \pi$,

$$\int_0^\infty \frac{\exp(-zx)}{\sqrt{x^2+bx}} \, dx = \exp\left(\frac{bz}{2}\right) K_0\left(\frac{bz}{2}\right) \tag{31}$$

(as usual, we take the principal branch of the square root function). A careful derivation of (31) can be found in Section 7.3.4 of [8] and it appears as Formula 3.383(8) in [10]. By combining (31) with the formula

$$K_n(z) = \frac{\pi}{2} \exp\left(n\frac{\pi}{2} i\right) i H_n(iz), \tag{32}$$

which can be found in Chapter VII of [8], we see that

$$\int_0^\infty \frac{\exp(-zx)}{\sqrt{x^2+bx}} \, dx = \frac{\pi}{2} i \exp\left(\frac{bz}{2}\right) H_0\left(i\frac{bz}{2}\right) \tag{33}$$

whenever $\mathrm{Re}(z) > 0$, $-\pi < \arg(bz) \le \pi/2$ and $|\arg(b)| < \pi$.

It is well known that for any complex numbers z and ν such that $\mathrm{Re}(\nu) > -1$ and $\mathrm{Re}(z) > 0$,

$$\int_0^\infty \exp(-zx) x^\nu \, dx = \frac{\Gamma(1+\nu)}{z^{1+\nu}}; \tag{34}$$

this identity appears, for instance, as Formula 3.381(4) in [10].

2.4. Steiltjes' asymptotic formula for Legendre functions

For all positive real ν and all $0 < \theta < \pi/2$ and positive integers M,

$$P_\nu(\cos(\theta)) = \left(\frac{2}{\pi\sin(\theta)}\right)^{1/2} \sum_{k=0}^{M-1} C_{\nu,k} \frac{\cos\left(\beta_{\nu,k}\right)}{\sin(\theta)^k} + R_{\nu,M}(\theta), \tag{35}$$

where

$$\beta_{\nu,k} = \left(\nu + k + \frac{1}{2}\right)\theta - \left(k + \frac{1}{2}\right)\frac{\pi}{2}, \tag{36}$$

$$C_{\nu,k} = \frac{\left(\Gamma\left(k+\frac{1}{2}\right)\right)^2 \Gamma(\nu+1)}{\pi 2^k \Gamma\left(\nu+k+\frac{3}{2}\right)\Gamma\left(k+1\right)}, \tag{37}$$

and

$$|R_{\nu,M}(\theta)| \le 2 \left(\frac{2}{\pi \sin(\theta)} \right)^{1/2} \frac{C_{\nu,M}}{\sin(\theta)^M}. \tag{38}$$

This approximation was introduced by Stieltjes; the error bound is a special case of Theorem 8.21.11 in Section 8.21 of [14]. In [1], an efficient method for computing the coefficients $C_{\nu,k}$ is suggested. The coefficient in the first term is given by

$$C_{\nu,0} = \frac{\Gamma\left(\nu+1\right)}{\Gamma\left(\nu+\frac{3}{2}\right)}. \tag{39}$$

This ratio of gamma functions can be approximated via a series in powers of $1/\nu$; however, it can be calculated more efficiently by observing that the related function

$$\tau(x) = \sqrt{x} \frac{\Gamma\left(x+\frac{1}{4}\right)}{\Gamma\left(x+\frac{3}{4}\right)} \tag{40}$$

admits an expansion in powers of $1/x^2$. In particular, the 7-term expansion

$$\begin{aligned} \tau(x) = 1 &- \frac{1}{64x^2} + \frac{21}{8192x^4} - \frac{671}{524288x^6} + \frac{180323}{134217728x^8} \\ &- \frac{20898423}{8589934592x^{10}} + \frac{7426362705}{1099511627776x^{12}} + O\left(\frac{1}{x^{14}}\right) \end{aligned} \tag{41}$$

gives roughly double precision accuracy for all $x > 10$ (see the discussion in [1]). The coefficient $C_{\nu,0}$ is related to τ through the formula

$$C_{\nu,0} = \frac{1}{\sqrt{\nu+\frac{3}{4}}} \tau\left(\nu + \frac{3}{4}\right). \tag{42}$$

The subsequent coefficients are obtained through the recurrence relation

$$C_{\nu,k+1} = \frac{\left(k+\frac{1}{2}\right)^2}{2(k+1)\left(\nu+k+\frac{3}{2}\right)} C_{\nu,k}. \tag{43}$$

2.5. Olver's uniform asymptotic expansion for φ_ν

By combining (10) and (11), we obtain the uniform asymptotic expansion

$$\varphi_\nu(\theta) \sim \sqrt{\lambda\theta} \left(H_0\left(\lambda\theta\right) \sum_{j=0}^{\infty} \frac{A_j(-\theta^2)}{\lambda^{2j}} - \frac{\theta}{\lambda} H_1\left(\lambda\theta\right) \sum_{j=0}^{\infty} \frac{B_j(-\theta^2)}{\lambda^{2j}} \right) \quad \text{as} \quad \nu \to \infty \tag{44}$$

of the function φ_ν defined via (24). Here, $\lambda = \nu + \frac{1}{2}$, as before. Only the first three coefficients in (44) can be expressed using elementary functions. They are:

$$A_0(\xi) = 1, \tag{45}$$

$$B_0(\xi) = \frac{\sqrt{-\xi}\,\mathrm{cotan}\left(\sqrt{-\xi}\right) - 1}{8\xi}, \tag{46}$$

and

$$A_1(\xi) = -\frac{\xi\,\mathrm{cotan}^2\left(\sqrt{-\xi}\right) - 6\sqrt{-\xi}\,\mathrm{cotan}\left(\sqrt{-\xi}\right) + 8\xi\,\mathrm{cosec}^2\left(\sqrt{-\xi}\right) + 15}{128\xi}. \tag{47}$$

Obviously, (44) can be rewritten as

$$\varphi_\nu(\theta) \sim \sqrt{\lambda\theta}\left(H_0\left(\lambda\theta\right)\sum_{j=0}^{\infty}\frac{\tilde{A}_j(\theta)}{\lambda^{2j}} - \frac{\theta}{\lambda}H_1\left(\lambda\theta\right)\sum_{j=0}^{\infty}\frac{\tilde{B}_j(\theta)}{\lambda^{2j}}\right) \quad \text{as} \quad \nu \to \infty \tag{48}$$

with

$$\tilde{A}_0(\xi) = 1, \tag{49}$$

$$\tilde{B}_0(\xi) = \frac{1-\xi\,\mathrm{cotan}(\xi)}{8\xi^2} \tag{50}$$

and

$$\tilde{A}_1(\xi) = \frac{15 - 6\xi\,\mathrm{cotan}(\xi) - \xi^2\,\mathrm{cotan}^2(\xi) - 8\xi^2\,\mathrm{cosec}^2(\xi)}{128\xi^2}. \tag{51}$$

As is clear from these formulas, when ξ is close to 0, the evaluation of $\tilde{B}_0(\xi)$ and $\tilde{A}_1(\xi)$ via (50) and (51) can lead to significant roundoff errors due to numerical cancellation.

3. An integral representation of a particular solution of Legendre's equation

In this section, we derive an integral representation of the function φ_ν defined via (24) involving only elementary functions. From (24), (29) and (30), we see that

$$\varphi_\nu(\theta) = \sqrt{\lambda\sin(\theta)}\frac{1}{\Gamma(\nu+1)}\int_0^\infty \exp(-\cos(\theta)t)H_0(\sin(\theta)t)t^\nu\;dt, \tag{52}$$

where $\lambda = \nu + \frac{1}{2}$, whenever $0 < \theta < \pi/2$ and $\mathrm{Re}(\nu) > -1$. We rearrange (52) as

$$\varphi_\nu(\theta) = \sqrt{\lambda\sin(\theta)}\frac{1}{\Gamma(\nu+1)}\int_0^\infty \exp(-\exp(-i\theta)t)\exp(-i\sin(\theta)t)H_0(\sin(\theta)t)t^\nu\;dt \tag{53}$$

and let $\beta = \exp(i\theta)\sin(\theta)$. Then, we apply Cauchy's theorem to change the contour of integration in (53) to that defined via

$$w(t) = \exp(-i\theta)t \quad \text{for all} \quad 0 < t < \infty. \tag{54}$$

This yields the formula

$$\varphi_\nu(\theta) = \sqrt{\lambda\sin(\theta)}\frac{\exp(i(\nu+1)\theta)}{\Gamma(\nu+1)}\int_0^\infty \exp(-w)\exp(-i\beta w)H_0(\beta w)w^\nu\;\;dw. \tag{55}$$

Now by letting $b = 2i$ and $z = \beta w$ in (33), we find that

$$-\frac{2}{\pi}i\int_0^\infty \frac{\exp(-\beta w x)}{\sqrt{x^2-2ix}}\;dx = \exp\left(-i\beta w\right)H_0\left(\beta w\right). \tag{56}$$

Formula (33) holds since $0 < \arg(z) < \pi/2$ and $\arg(b) = -\pi/2$. Inserting (56) into (55) yields

$$\varphi_\nu(\theta) = -i\frac{2}{\pi}\sqrt{\lambda\sin(\theta)}\frac{\exp(i(\nu+1)\theta)}{\Gamma(\nu+1)}\int_0^\infty\int_0^\infty \frac{\exp(-(1+\beta x)w)}{\sqrt{x^2-2ix}}w^\nu\;dx\;dw. \tag{57}$$

Since $\mathrm{Re}(1+\beta x) > 0$, we may use Formula (34) to evaluate the integral with respect to w in (57) whenever $\mathrm{Re}(\nu) > -1$. In this way we see that

$$\varphi_\nu(\theta) = -i\frac{2}{\pi}\sqrt{\lambda\sin(\theta)}\exp(i(\nu+1)\theta)\int_0^\infty \frac{1}{\sqrt{x^2-2ix}\,(1+\beta x)^{\nu+1}}\;dx \tag{58}$$

when $\mathrm{Re}(\nu) > -1$ and $0 < \theta < \pi/2$. Finally, we invoke Cauchy's theorem once more to change the contour of integration in (58) to that defined via

$$\tau(x) = \beta x \quad \text{for all} \quad 0 < x < \infty. \tag{59}$$

This yields

$$\varphi_\nu(\theta) = -i\frac{2}{\pi}\sqrt{\lambda \sin(\theta)}\exp(i(\nu+1)\theta)\int_0^\infty \frac{1}{\sqrt{\tau^2 - 2i\beta\tau}\,(1+\tau)^{\nu+1}}\,d\tau \tag{60}$$

for all ν and θ such that $\mathrm{Re}(\nu) > -1$ and $0 < \theta < \pi/2$. From (60), we see that the function F_ν appearing in (25) is

$$F_\nu(\theta) = \int_0^\infty \frac{1}{\sqrt{\tau^2 - 2i\beta\tau}\,(1+\tau)^{\nu+1}}\,d\tau. \tag{61}$$

One of the principal differences between the expression (60) and Olver's asymptotic expansion (48) is that because $|\exp(i(\nu+1)\theta)| = 1$, (60) immediately gives yields an expression for $|\varphi_\nu|^2$ which obviously involves only nonoscillatory functions. In particular,

$$|\varphi_\nu(\theta)|^2 = \frac{4}{\pi}\left(\nu + \frac{1}{2}\right)\sin(\theta)\,|F_\nu(\theta)|^2\,. \tag{62}$$

On the other hand, applying the same procedure to a n-term truncation of the expansion (48) when $n > 1$ yields a complicated expression consisting of a sum of products of oscillatory functions which happen to cancel in such a way that the result is nonoscillatory. This observation is of interest since $\alpha_\nu'(\theta)$ is related to $|\varphi_\nu(\theta)|^2$ through the formula

$$\alpha_\nu'(\theta) = \frac{2}{\pi}\left(\nu + \frac{1}{2}\right)|\varphi_\nu(\theta)|^{-2}\,, \tag{63}$$

as is obvious from (7), (9) and (24).

4. An asymptotic formula for Legendre functions of large degrees

In order to derive an asymptotic expansion for φ_ν, we replace the function

$$f(\tau) = \frac{1}{(1+\tau)^{\nu+1}} \tag{64}$$

in (60) with a sum of the form

$$g(\tau) = a_0\exp(-p\tau) + \sum_{k=1}^{N} a_k \exp(-(p+kq)\tau) + b_k\exp(-(p-kq)\tau), \tag{65}$$

where $p = \nu + 1$, $q = \sqrt{p}$, and the coefficients $a_0, a_1, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ are chosen so the power series expansions of f and g around 0 agree to order $2N$. That is, we require that the system of $2N + 1$ linear equations

$$f^{(k)}(0) = g^{(k)}(0) \quad \text{for all} \quad k = 0, 1, \ldots, 2N \tag{66}$$

in the $2N+1$ variables $a_0, a_1, \dots, a_N, b_1, b_2, \dots, b_N$ be satisfied. By so doing, we obtain the approximation

$$\begin{aligned}\varphi_\nu(\theta) \approx -\frac{2}{\pi} i \exp(i(\nu+1)\theta)\sqrt{\lambda \sin(\theta)} \Bigg(a_0 \int_0^\infty \frac{\exp(-p\tau)}{\sqrt{\tau^2 - 2i\beta}}\ d\tau \\ + \sum_{k=1}^N a_k \int_0^\infty \frac{\exp(-\left(p+kq\right)\tau)}{\sqrt{\tau^2-2i\beta}}\ d\tau + \sum_{k=1}^N b_k \int_0^\infty \frac{\exp(-\left(p-kq\right)\tau)}{\sqrt{\tau^2-2i\beta}}\ d\tau \Bigg).\end{aligned} \tag{67}$$

Now by applying Formula (33) to (67) we conclude that

$$\begin{aligned}\varphi_\nu(\theta) \approx \exp(i(\nu+1)\theta)\sqrt{\lambda\sin(\theta)}\Bigg(a_0 \exp(-i\beta p) H_0(\beta p) + \\ \sum_{k=1}^N a_k \exp(-i\beta\left(p+kq\right)) H_0\left(\beta\left(p+kq\right)\right) + b_k \exp(-i\beta\left(p-kq\right)) H_0\left(\beta\left(p-kq\right)\right)\Bigg),\end{aligned} \tag{68}$$

where $\beta = \sin(\theta)\exp(i\theta)$, $p = \nu+1$, and $q = \sqrt{p}$. The use of Formula (33) is justified so long as $\mathrm{Re}(p) > N^2$ and $0 < \theta < \pi/2$ (the second condition ensures that $|\arg(2i\beta)| < \pi$). When $N=0$, Formula (68) becomes

$$\varphi_\nu(\theta) \approx \exp(i(\nu+1)\theta)\sqrt{\lambda\sin(\theta)}\exp(-i\beta p) H_0(\beta p). \tag{69}$$

For larger values of N the linear system (66) can be solved easily using a computer algebra system. Our Mathematica script for doing so appears in an appendix of this paper. A second appendix lists the coefficients when $N=1$, $N=2$, $N=3$, $N=4$, $N=5$ and $N=6$. That the functions

$$\exp\left(-i\beta\left(p \pm kq\right)\right) H_0\left(\beta\left(p \pm kq\right)\right) \tag{70}$$

appearing in (68) are nonoscillatory is obvious from the formula

$$\exp\left(-iz\right) H_0\left(z\right) = -\frac{2}{\pi} i \int_0^\infty \frac{\exp(-zx)}{\sqrt{x^2-2ix}}\ dx \tag{71}$$

obtained by letting $b = -2i$ in (33).

One of the most useful features of (68) is that it gives an asymptotic expansion of α_ν' which does not involve oscillatory functions. In particular, it follows immediately by taking absolute values in (68) and making use of (63) that

$$\begin{aligned}\alpha_\nu'(\theta) \approx \frac{2}{\pi\sin(\theta)} \Bigg| a_0 \exp(-i\beta p) H_0(\beta p) + \\ \sum_{k=1}^N a_k \exp(-i\beta\left(p+kq\right)) H_0\left(\beta\left(p+kq\right)\right) + b_k \exp(-i\beta\left(p-kq\right)) H_0\left(\beta\left(p-kq\right)\right)\Bigg|^{-2},\end{aligned} \tag{72}$$

where $\beta = \sin(\theta)\exp(i\theta)$, $p = \nu+1$, and $q = \sqrt{p}$.

Errors bounds for the asymptotic expansion (68) can be obtained in a straightforward fashion; however, the calculations are somewhat tedious. Here, we give the details only in the case $N=0$.

We first observe that

$$\left|\frac{1}{\sqrt{\tau^2-2i\beta\tau}}\frac{1}{(1+\tau)^p}-\frac{1}{\sqrt{\tau^2-2i\beta\tau}}\exp(-p\tau)\right| \leq \frac{1}{\tau}\left(\frac{1}{(1+\tau)^p}-\exp(-p\tau)\right) \tag{73}$$

for all $p>0$, $0<\theta<\frac{\pi}{2}$ and $\tau>0$. It follows that

$$\int_\epsilon^\infty \left|\frac{1}{\sqrt{\tau^2-2i\beta\tau}}\frac{1}{(1+\tau)^p}-\frac{1}{\sqrt{\tau^2-2i\beta\tau}}\exp(-p\tau)\right|\,d\tau \leq \int_\epsilon^\infty \frac{1}{\tau}\left(\frac{1}{(1+\tau)^p}-\exp(-(\nu+1)\tau)\right)\,d\tau \tag{74}$$

for all $p>0$, $\epsilon>0$ and $0<\theta<\frac{\pi}{2}$. Now

$$\int_\epsilon^\infty\left(\frac{1}{\tau}\frac{1}{(1+\tau)^p}\ -\frac{1}{\tau}\exp(-p\tau)\right)d\tau=\frac{\epsilon^{-p}}{p}{}_2F_1\left(p,p;p+1;-\frac{1}{\epsilon}\right)-\Gamma(0,p\epsilon), \tag{75}$$

where ${}_2F_1\left(p,p;p+1;-\frac{1}{\epsilon}\right)$ denotes the hypergeometric series

$${}_2F_1\left(p,p;p+1;-\frac{1}{\epsilon}\right)=\sum_{n=0}^\infty(-1)^n\frac{(\Gamma(p+n))^2\,\Gamma(p+1)}{(\Gamma(p))^2\,\Gamma(p+n+1)}\frac{1}{\epsilon^n\Gamma(n+1)} \tag{76}$$

and $\Gamma(z,a)$ is the upper incomplete Gamma function. Formula (75) can be verified directly by differentiation. From transformation formulas for Gauss' hypergeometric function (see, for instance, Section 2.9 of [9]), it follows that

$$\frac{\epsilon^{-p}}{p}{}_2F_1\left(p,p;p+1;-\frac{1}{\epsilon}\right)=-\gamma-\log(\epsilon)-\psi(p)+\mathcal{O}\left(\epsilon\right). \tag{77}$$

Here, we use ψ to denote the digamma function

$$\psi(z)=\frac{d}{dz}\log\left(\Gamma(z)\right) \tag{78}$$

and γ is Euler's constant. Similarly, $\Gamma(0,\epsilon p)$ admits the expansion

$$\Gamma(0,\epsilon p)=-\gamma-\log(p)-\log(\epsilon)+\mathcal{O}\left(\epsilon\right) \tag{79}$$

(see, for instance, Section 8.5 of [10]). By combining (75) with (77) and (79), we see that

$$\int_\epsilon^\infty\left(\frac{1}{\tau}\frac{1}{(1+\tau)^p}\ -\frac{1}{\tau}\exp(-p\tau)\right)d\tau=\log(p)-\psi(p)+\mathcal{O}\left(\epsilon\right). \tag{80}$$

By taking the limit of both sides of (80) as $\epsilon\to0$, we obtain

$$\int_0^\infty\left(\frac{1}{\tau}\frac{1}{(1+\tau)^p}\ -\frac{1}{\tau}\exp(-p\tau)\right)d\tau=\log(p)-\psi(p). \tag{81}$$

It is the case that

$$0\leq\log(p)-\psi(p)\leq\frac{1}{p} \tag{82}$$

for all $p>0$; a proof of this can be found, for instance, in [5]. It follows from (73), (81) and (82) that

$$\left| \int_0^\infty \left(\frac{1}{\sqrt{\tau^2 - 2i\beta\tau}} \frac{1}{(1+\tau)^p} - \frac{1}{\sqrt{\tau^2 - 2i\beta\tau}} \exp(-p\tau) \right) \, d\tau \right| \leq \frac{1}{p}. \tag{83}$$

We combine (83) with (60) and (33) to conclude that

$$\left| \varphi_\nu(\theta) - \exp(i(\nu+1)\theta)\sqrt{\lambda \sin(\theta)} \exp(-i\beta p) H_0(\beta p) \right| \leq \frac{2}{\pi} \frac{1}{p} \tag{84}$$

for all $\nu > 0$ and $0 < \theta < \frac{\pi}{2}$. We note that the bound (84) is uniform in θ for all $0 < \theta < \frac{\pi}{2}$.

5. Numerical experiments

In this section, we describe numerical experiments which were conducted to assess the performance of the asymptotic expansions of Section 4. Our code was written in Fortran and compiled with the GNU Fortran Compiler version 5.2.1. The calculations were carried out on a laptop equipped with an Intel i7-5600U processor running at 2.60GHz.

5.1. The accuracy of the expansion (68) as a function of N and ν

We measured the accuracy of the expansion (68) for various values of N and ν. For each pair of values of N and ν considered, we evaluated (68) at a collection of 1,000 points on the interval $(0, \pi/2)$. The first 500 points were drawn at random from the uniform distribution on the interval

ν	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
10^2	1.55×10^{-06}	5.30×10^{-08}	1.48×10^{-09}	6.05×10^{-11}	1.17×10^{-11}
$10^2\pi$	5.02×10^{-08}	5.00×10^{-10}	2.84×10^{-12}	6.49×10^{-14}	5.63×10^{-14}
10^3	1.55×10^{-09}	4.74×10^{-12}	2.09×10^{-13}	2.09×10^{-13}	2.09×10^{-13}
$10^3\pi$	5.02×10^{-11}	1.16×10^{-12}	1.16×10^{-12}	1.16×10^{-12}	1.16×10^{-12}
10^4	2.46×10^{-12}	1.90×10^{-12}	1.90×10^{-12}	1.90×10^{-12}	1.90×10^{-12}
$10^4\pi$	6.70×10^{-12}	6.70×10^{-12}	6.70×10^{-12}	6.70×10^{-12}	6.70×10^{-12}
10^5	2.17×10^{-11}	2.17×10^{-11}	2.17×10^{-11}	2.17×10^{-11}	2.17×10^{-11}
$10^5\pi$	1.11×10^{-10}	1.11×10^{-10}	1.11×10^{-10}	1.11×10^{-10}	1.11×10^{-10}
10^6	2.15×10^{-10}	2.15×10^{-10}	2.15×10^{-10}	2.15×10^{-10}	2.15×10^{-10}
$10^6\pi$	9.36×10^{-10}	9.36×10^{-10}	9.36×10^{-10}	9.36×10^{-10}	9.36×10^{-10}
10^7	2.00×10^{-09}	2.00×10^{-09}	2.00×10^{-09}	2.00×10^{-09}	2.00×10^{-09}
$10^7\pi$	7.87×10^{-09}	7.87×10^{-09}	7.87×10^{-09}	7.87×10^{-09}	7.87×10^{-09}
10^8	2.33×10^{-08}	2.33×10^{-08}	2.33×10^{-08}	2.33×10^{-08}	2.33×10^{-08}
$10^8\pi$	1.06×10^{-07}	1.06×10^{-07}	1.06×10^{-07}	1.06×10^{-07}	1.06×10^{-07}
10^9	2.15×10^{-07}	2.15×10^{-07}	2.15×10^{-07}	2.15×10^{-07}	2.15×10^{-07}

Table 1: The relative accuracy of the expansion (68) as a function of ν and N. Here, the expansion was evaluated using double precision arithmetic.

$(0, \pi/2)$, while the remaining points were constructed by drawing 500 points from the uniform distribution on the interval $(0, 1)$ and applying the mapping $t \to \exp(-36t)$ to each of them. In this way, we ensured that the accuracy of (68) was tested near the singular point of Legendre's equation which occurs when $\theta = 0$.

The results are reported in Tables 1 and 2. Table 1 gives the maximum relative error in the value of φ_ν which was observed when these calculations were carried out in double precision arithmetic as a function of ν and N, and Table 2 reports the maximum relative error in the value of φ_ν which was observed when they were performed using quadruple precision arithmetic as a function of ν and N. Reference values were computed by running the algorithm of [2] in quadruple precision arithmetic. Like the asymptotic expansion (68), the algorithm of [2] is capable of achieving high accuracy near the singular point of Legendre's equation whereas Stieltjes' expansion (35) and the well-known three term recurrence relations are inaccurate when θ is near 0. See [1, 11], though, for the derivation of asymptotic expansions of Legendre functions which are accurate for θ near 0 and [13, 6, 7] for expansions of Legendre functions which are accurate for all $\theta \in (0, \pi/2)$.

The routine we used to evaluate the Hankel function of order 0 achieves roughly double precision accuracy, even when executed using quadruple precision arithmetic. Thus the minimum error achieved when the computations were performed using quadruple precision arithmetic was on the order of 10^{-16} (see Table 2).

We also observe that relative accuracy was lost as ν increases when (68) is evaluated using double precision arithmetic. This loss of precision is unsurprising and consistent with the condition

ν	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$
10^2	1.55×10^{-06}	5.30×10^{-08}	1.48×10^{-09}	6.05×10^{-11}	1.17×10^{-11}
$10^2\pi$	5.02×10^{-08}	5.00×10^{-10}	2.84×10^{-12}	3.28×10^{-14}	3.15×10^{-15}
10^3	1.55×10^{-09}	4.74×10^{-12}	7.06×10^{-15}	3.94×10^{-16}	4.10×10^{-16}
$10^3\pi$	5.02×10^{-11}	4.83×10^{-14}	3.72×10^{-16}	3.58×10^{-16}	3.61×10^{-16}
10^4	1.55×10^{-12}	4.80×10^{-16}	4.94×10^{-16}	4.70×10^{-16}	4.73×10^{-16}
$10^4\pi$	5.02×10^{-14}	4.95×10^{-16}	5.06×10^{-16}	4.84×10^{-16}	4.87×10^{-16}
10^5	1.53×10^{-15}	3.41×10^{-16}	3.50×10^{-16}	3.28×10^{-16}	3.31×10^{-16}
$10^5\pi$	2.31×10^{-16}	2.39×10^{-16}	2.49×10^{-16}	2.24×10^{-16}	2.27×10^{-16}
10^6	5.10×10^{-16}	4.86×10^{-16}	4.95×10^{-16}	4.73×10^{-16}	4.76×10^{-16}
$10^6\pi$	4.89×10^{-16}	4.64×10^{-16}	4.72×10^{-16}	4.51×10^{-16}	4.54×10^{-16}
10^7	4.85×10^{-16}	4.57×10^{-16}	4.67×10^{-16}	4.42×10^{-16}	4.45×10^{-16}
$10^7\pi$	5.17×10^{-16}	4.90×10^{-16}	4.99×10^{-16}	4.75×10^{-16}	4.78×10^{-16}
10^8	4.43×10^{-16}	4.14×10^{-16}	4.24×10^{-16}	3.98×10^{-16}	4.02×10^{-16}
$10^8\pi$	3.91×10^{-16}	3.64×10^{-16}	3.73×10^{-16}	3.50×10^{-16}	3.53×10^{-16}
10^9	4.91×10^{-16}	4.63×10^{-16}	4.72×10^{-16}	4.48×10^{-16}	4.51×10^{-16}

Table 2: The relative accuracy of the expansion (68) as a function of ν and N. Here, the expansion was evaluated using quadruple precision arithmetic.

number of the evaluation of the highly oscillatory functions P_ν and Q_ν. Indeed, evaluating $P_\nu(x)$ and $Q_\nu(x)$ is analogous to calculating the values of $\cos(\nu x)$ and $\sin(\nu x)$. If $\cos(\nu x)$ and $\sin(\nu x)$ can be evaluated with accuracy on the order of ϵ, then so can the value of $\mathrm{mod}(\nu, 1)$. This clearly limits the precision with which $\cos(\nu x)$ and $\sin(\nu x)$ can be evaluated using finite precision arithmetic. A similar argument applies to $P_\nu(x)$ and $Q_\nu(x)$. The only exception is when the value of $\mathrm{mod}(\nu, 1)$ is known to high precision (for instance, when ν is an integer). Then $\cos(\nu x)$ and $\sin(\nu x)$ can be calculated with comparable precision, as can P_ν and Q_ν (via the well-known three term recurrence relations, for instance).

5.2. The accuracy of the expansion (68) as a function of θ

In order to measure the accuracy of the expansion (68) as a function of θ, we sampled a collection of 1,000 points on the interval $(0, \pi/2)$ using the same method as in Section 5.1. Then, we evaluated (68) with $N = 2$ and $\nu = 1,000$ at each of the chosen points. We repeated this procedure with N taken to be 3 and $\nu = 1,000$, and with N taken to be 4 and $\nu = 1,000$. The base-10 logarithms of the resulting relative errors are plotted in Figure 3. Once again, reference values were computed by running the algorithm of [2] in quadruple precision arithmetic. We observe that the error in Formula (68) is remarkably uniform as a function of θ — it is nearly constant until θ nears the singular point at 0, at which point it decreases slightly.

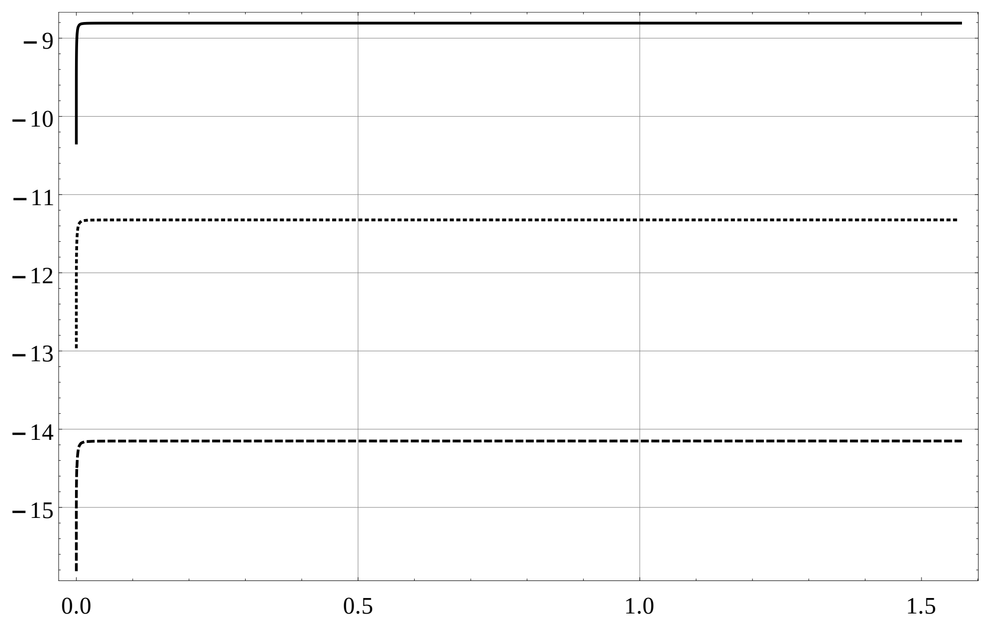

Figure 3: The base-10 logarithm of the relative accuracy of the expansion (68) as a function of θ when $\nu = 1{,}000$ and $N = 2$ (top line), $N = 3$ (middle line), $N = 4$ (bottom line).

5.3. The speed of the expansion (68) as function of N

Next, we measured the time required to evaluate φ_ν using the expansion (68). In particular, for several pairs of values of N and ν, we evaluated (68) at a collection of 1,000 points drawn from the uniform distribution on the interval $(0, \pi/2)$. We also applied the same procedure to

Stieltjes' expansion (35) with $M = 16$. Table 3 gives the average time required to evaluate (68) and (35).

We note that while the time required to evaluate (68) is slightly larger than the time required to evaluate Stieltjes' Formula (35), the expansion of Section 4 gives both the value of P_ν and that of Q_ν while Stieltjes' formula gives only the value of P_ν. Moreover, unlike Stieltjes' formula, our asymptotic approximation is accurate for θ near 0.

ν	$N = 3$	$N = 4$	$N = 5$	$N = 6$	Stieltjes' formula
10^2	1.49×10^{-06}	1.83×10^{-06}	2.41×10^{-06}	2.67×10^{-06}	1.69×10^{-06}
$10^2\pi$	1.42×10^{-06}	1.77×10^{-06}	2.40×10^{-06}	2.44×10^{-06}	1.60×10^{-06}
10^3	1.35×10^{-06}	1.73×10^{-06}	2.34×10^{-06}	2.38×10^{-06}	1.59×10^{-06}
$10^3\pi$	1.41×10^{-06}	1.65×10^{-06}	2.36×10^{-06}	2.42×10^{-06}	1.62×10^{-06}
10^4	1.32×10^{-06}	1.77×10^{-06}	2.36×10^{-06}	2.44×10^{-06}	1.61×10^{-06}
$10^4\pi$	1.31×10^{-06}	1.65×10^{-06}	2.36×10^{-06}	2.37×10^{-06}	1.57×10^{-06}
10^5	1.33×10^{-06}	1.73×10^{-06}	2.36×10^{-06}	2.35×10^{-06}	1.56×10^{-06}
$10^5\pi$	1.31×10^{-06}	1.72×10^{-06}	2.39×10^{-06}	2.42×10^{-06}	1.58×10^{-06}
10^6	1.31×10^{-06}	1.68×10^{-06}	2.39×10^{-06}	2.35×10^{-06}	1.56×10^{-06}
$10^6\pi$	1.33×10^{-06}	1.66×10^{-06}	2.37×10^{-06}	2.38×10^{-06}	1.55×10^{-06}
10^7	1.39×10^{-06}	1.67×10^{-06}	2.35×10^{-06}	2.41×10^{-06}	1.58×10^{-06}
$10^7\pi$	1.32×10^{-06}	1.65×10^{-06}	2.32×10^{-06}	2.40×10^{-06}	1.58×10^{-06}
10^8	1.31×10^{-06}	1.70×10^{-06}	2.33×10^{-06}	2.48×10^{-06}	1.60×10^{-06}
$10^8\pi$	1.35×10^{-06}	1.68×10^{-06}	2.36×10^{-06}	2.41×10^{-06}	1.83×10^{-06}
10^9	1.37×10^{-06}	1.72×10^{-06}	2.33×10^{-06}	2.39×10^{-06}	1.69×10^{-06}

Table 3: A comparison of the average time (in seconds) required to evaluate (68) for various values of N and ν with the time requires to evaluate the first 16 terms of Stieltjes' expansion (35). Note that Stieltjes' formula yields only the value of P_ν while (68) yields both P_ν and Q_ν. Moreover, unlike (68), Stieltjes' is not accurate for arguments close to the singular points of Legendre's differential equation.

5.4. The accuracy of the expansion (72)

In this experiment, we measured the accuracy achieved by the expansion (72). More specifically, for each of several pairs of values of N and ν, we evaluated (72) at a collection of 1,000 points in the interval $(0, \pi/2)$ which were chosen as in Section 5.1. These experiments were performed using double precision arithmetic. The obtained values of α'_ν were compared with reference values computed by running the algorithm of [2] using quadruple precision arithmetic.

The results are reported in Table 4. For each pair of values of ν and N, it lists the maximum relative error in α'_ν which was observed. We note that, unlike the experiments of Section 5.1, near double precision accuracy was obtained by performing the calculations in double precision arithmetic. This is not surprising since the condition number of the evaluation of the nonoscillatory function α'_ν is small and not dependent on ν.

ν	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$
10^2	4.87×10^{-07}	2.07×10^{-08}	7.36×10^{-09}	7.04×10^{-09}	7.03×10^{-09}
$10^2\pi$	1.53×10^{-08}	1.64×10^{-10}	1.60×10^{-10}	1.60×10^{-10}	1.60×10^{-10}
10^3	4.78×10^{-10}	1.22×10^{-12}	2.42×10^{-15}	1.27×10^{-15}	1.41×10^{-15}
$10^3\pi$	1.54×10^{-11}	1.29×10^{-14}	1.07×10^{-15}	1.27×10^{-15}	1.36×10^{-15}
10^4	4.78×10^{-13}	1.36×10^{-15}	1.36×10^{-15}	1.48×10^{-15}	1.36×10^{-15}
$10^4\pi$	1.56×10^{-14}	1.08×10^{-15}	1.26×10^{-15}	1.08×10^{-15}	1.34×10^{-15}
10^5	1.10×10^{-15}	9.65×10^{-16}	1.38×10^{-15}	1.14×10^{-15}	1.20×10^{-15}
$10^5\pi$	9.89×10^{-16}	8.95×10^{-16}	1.04×10^{-15}	1.00×10^{-15}	1.35×10^{-15}
10^6	1.30×10^{-15}	1.30×10^{-15}	1.19×10^{-15}	1.30×10^{-15}	1.33×10^{-15}
$10^6\pi$	1.39×10^{-15}	1.39×10^{-15}	1.25×10^{-15}	1.25×10^{-15}	1.37×10^{-15}
10^7	1.22×10^{-15}	1.22×10^{-15}	1.22×10^{-15}	1.70×10^{-15}	1.62×10^{-15}
$10^7\pi$	1.44×10^{-15}	1.44×10^{-15}	1.44×10^{-15}	1.55×10^{-15}	1.44×10^{-15}
10^8	1.56×10^{-15}	1.42×10^{-15}	1.45×10^{-15}	1.56×10^{-15}	1.42×10^{-15}
$10^8\pi$	9.47×10^{-16}	9.75×10^{-16}	1.04×10^{-15}	1.10×10^{-15}	1.33×10^{-15}
10^9	1.12×10^{-15}	1.34×10^{-15}	1.34×10^{-15}	1.41×10^{-15}	1.38×10^{-15}

Table 4: The relative accuracy with which the derivative of the nonoscillatory phase function for Legendre's differential equation is evaluated via the expansions of Section 4. These calculations were performed using double precision arithmetic.

5.5. Comparison with Olver's asymptotic expansion

In this experiment, we repeated the procedure of preceding section, but this time we used the combination of Olver's asymptotic expansion (48) and Formula (63) to evaluate α'_ν. We calculated the coefficients in Olver's expansion through (49), (50) and (51). In some cases, accuracy was lost due to the numerical cancellation errors which occur when these formulas are evaluated in a straightforward fashion. Table 5 shows the results. There, the accuracies obtained using the first 1, 2 and 3 terms of Olver's expansions are given as a function of ν.

We observe that Olver's expansions are generally more efficient than those of Section 4. However, Tables 4 and 5 also reveal the two principal advantages of the asymptotic expansions of Section 4. First, since their coefficients are easy to calculate, high order expansions can be used with the consequence that higher accuracy approximations can sometimes be obtained via (72) than using Olver's expansions. Second, and more critically, the expansions of Section 4 do not suffer from the difficulties with numerical cancellation which limit the precision obtained by Olver's asymptotic expansion when its coefficients are evaluated in a straightforward fashion.

6. Conclusions

Nonoscillatory phase functions are powerful analytic and numerical tools. Among other things, explicit formulas for them can be used to efficiently and accurately evaluate special functions, their zeros and to apply special function transforms.

ν	One term	Two terms	Three terms
10^2	7.34×10^{-06}	2.95×10^{-06}	2.07×10^{-05}
$10^2\pi$	7.49×10^{-07}	3.02×10^{-07}	2.10×10^{-06}
10^3	7.41×10^{-08}	2.98×10^{-08}	2.07×10^{-07}
$10^3\pi$	7.52×10^{-09}	3.03×10^{-09}	2.08×10^{-08}
10^4	7.42×10^{-10}	2.99×10^{-10}	2.04×10^{-09}
$10^4\pi$	7.52×10^{-11}	3.03×10^{-11}	2.05×10^{-10}
10^5	7.42×10^{-12}	2.99×10^{-12}	2.00×10^{-11}
$10^5\pi$	7.52×10^{-13}	3.03×10^{-13}	1.99×10^{-12}
10^6	7.41×10^{-14}	3.34×10^{-14}	1.93×10^{-13}
$10^6\pi$	7.85×10^{-15}	3.91×10^{-15}	1.90×10^{-14}
10^7	1.11×10^{-15}	8.24×10^{-16}	2.05×10^{-15}
$10^7\pi$	8.30×10^{-16}	8.30×10^{-16}	8.30×10^{-16}
10^8	8.51×10^{-16}	8.51×10^{-16}	8.51×10^{-16}
$10^8\pi$	9.48×10^{-16}	9.48×10^{-16}	9.48×10^{-16}
10^9	7.15×10^{-16}	7.15×10^{-16}	7.15×10^{-16}

Table 5: The relative accuracy with which the derivative of the nonoscillatory phase function for Legendre's differential equation is evaluated via Olver's uniform asymptotic expansion (48). These calculations were performed using double precision arithmetic.

Here, we factored a particular solution of Legendre's differential equation as the product of an oscillatory exponential function and an integral involving only nonoscillatory elementary functions. By so doing, we showed the existence of a nonoscillatory phase function and derived an asymptotic formula a type which is useful for evaluating the derivative of the nonoscillatory phase function.

We will report on the use of the results of the paper to apply the Legendre transform rapidly and on generalizations of this work to the case of associated Legendre functions, prolate spheroidal wave functions and other related special functions at a later date.

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Appendix A. Mathematica source code

Below is a Mathematica script for generating the coefficients $a_0, a_1, b_1, \ldots, a_n, b_n$ in the expansion (68).

```
(*********************************************************************************)
ClearAll["Global`*"]

(* Construct the (2n+1)-term expansion *)
n=3;
f[t_]=1/(1+t)^(q^2);
g[t_]=a0*Exp[-q^2*t]+Sum[a[k]*Exp[-(q^2+k*q)*t]+b[k]*Exp[-(q^2-k*q)*t],{k,1,n}];

eqs={};
vars={};
h0[t_]=f[t]-g[t];
For[k=0,k<=2*n,k++,eqs=Simplify[Join[eqs,{h0[0]==0}]];h0[t_]=h0'[t]];
For[k=1,k<=n,k++,vars=Join[vars,{a[k],b[k]}]];
vars=Join[vars,{a0}];
sol=Simplify[Solve[eqs,vars][[1]],p>0]

(* Write the TEX version of the expansion to the file tmp2 *)
OpenWrite["tmp2"];
WriteString["tmp2","\\begin{equation}\n\\begin{aligned}\n"];
WriteString["tmp2","a_0 &="ToString[TeXForm[a0/.sol]],", \\\\ \n"];
For[k=1,k<=n,k++,
WriteString["tmp2","a_",k," &="ToString[TeXForm[a[k]/.sol]],", \\\\ \n"]];
For[k=1,k<=n-1,k++,
WriteString["tmp2","b_",k," &="ToString[TeXForm[b[k]/.sol]],", \\\\ \n"]];
WriteString["tmp2","b_",k," &="ToString[TeXForm[b[k]/.sol]],".\n"];
WriteString["tmp2","\\end{aligned}\n\\end{equation}"];
Close["tmp2"];

(* Write the FORTRAN version of the expansion to tmp *)
OpenWrite["tmp"];
WriteString["tmp","a0 = ",ToString[FortranForm[N[Expand[a0/.sol],36]]],"\n"];
For[k=1,k<=n,k++,
WriteString["tmp","a(",k,") ="ToString[FortranForm[N[Expand[a[k]/.sol],36]]],"\n"]];
For[k=1,k<=n-1,k++,
WriteString["tmp","b(",k,") ="ToString[FortranForm[N[Expand[b[k]/.sol],36]]],"\n"]];
WriteString["tmp","b(",k,") ="ToString[FortranForm[N[Expand[b[k]/.sol],36]]],"\n"];
Close["tmp"];
(*********************************************************************************)
```

Appendix B. The coefficients in the expansion (68)

When $N = 1$, the coefficients are

$$a_0 = 0,$$
$$a_1 = \frac{1}{2},$$
$$b_1 = \frac{1}{2}.$$

When $N = 2$, the coefficients are

$$a_0 = \frac{3}{2q^2} + \frac{1}{2},$$
$$a_1 = \frac{q^2 - 2q - 6}{6q^2},$$
$$a_2 = \frac{q^2 + 2q + 3}{12q^2},$$
$$b_1 = \frac{q^2 + 2q - 6}{6q^2},$$
$$b_2 = \frac{q^2 - 2q + 3}{12q^2}.$$

When $N = 3$, they are

$$a_0 = -\frac{-7q^4 + 23q^2 + 60}{18q^4},$$
$$a_1 = \frac{6q^4 - 3q^3 + 26q^2 + 12q + 60}{24q^4},$$
$$a_2 = -\frac{-3q^4 + 35q^2 + 24q + 60}{60q^4},$$
$$a_3 = \frac{2q^4 + 15q^3 + 50q^2 + 36q + 60}{360q^4},$$
$$b_1 = \frac{6q^4 + 3q^3 + 26q^2 - 12q + 60}{24q^4},$$
$$b_2 = \frac{3q^4 - 35q^2 + 24q - 60}{60q^4},$$
$$b_3 = \frac{2q^4 - 15q^3 + 50q^2 - 36q + 60}{360q^4}.$$

When $N = 4$:

$$a_0 = \frac{115q^6 + 59q^4 + 1854q^2 + 2520}{288q^6},$$
$$a_1 = -\frac{-87q^6 + 59q^5 + 37q^4 + 114q^3 + 1914q^2 + 360q + 2520}{360q^6},$$
$$a_2 = \frac{39q^6 + 28q^5 + 7q^4 + 300q^3 + 2094q^2 + 720q + 2520}{720q^6},$$
$$a_3 = -\frac{-11q^6 - 63q^5 + 77q^4 + 630q^3 + 2394q^2 + 1080q + 2520}{2520q^6},$$
$$a_4 = \frac{3q^6 + 56q^5 + 427q^4 + 1176q^3 + 2814q^2 + 1440q + 2520}{20160q^6},$$
$$b_1 = \frac{87q^6 + 59q^5 - 37q^4 + 114q^3 - 1914q^2 + 360q - 2520}{360q^6},$$
$$b_2 = \frac{39q^6 - 28q^5 + 7q^4 - 300q^3 + 2094q^2 - 720q + 2520}{720q^6},$$
$$b_3 = -\frac{-11q^6 + 63q^5 + 77q^4 - 630q^3 + 2394q^2 - 1080q + 2520}{2520q^6},$$
$$b_4 = \frac{3q^6 - 56q^5 + 427q^4 - 1176q^3 + 2814q^2 - 1440q + 2520}{20160q^6}.$$

When $N = 5$:

$$
\begin{aligned}
a_0 &= -\frac{-359q^8 + 20q^6 + 1530q^4 + 21636q^2 + 22680}{900q^8},\\
a_1 &= \frac{2091q^8 - 1396q^7 + 747q^6 - 104q^5 + 12654q^4 + 12672q^3 + 175608q^2 + 20160q + 181440}{8640q^8},\\
a_2 &= -\frac{-204q^8 - 137q^7 + 372q^6 - 259q^5 + 3654q^4 + 6876q^3 + 45792q^2 + 10080q + 45360}{3780q^8},\\
a_3 &= \frac{179q^8 + 1068q^7 + 403q^6 - 2184q^5 + 20286q^4 + 46656q^3 + 195768q^2 + 60480q + 181440}{40320q^8},\\
a_4 &= -\frac{-3q^8 - 53q^7 - 276q^6 - 7q^5 + 4158q^4 + 9036q^3 + 26676q^2 + 10080q + 22680}{22680q^8},\\
a_5 &= \frac{3q^8 + 100q^7 + 1635q^6 + 13160q^5 + 58590q^4 + 106560q^3 + 236088q^2 + 100800q + 181440}{1814400q^8},\\
b_1 &= \frac{2091q^8 + 1396q^7 + 747q^6 + 104q^5 + 12654q^4 - 12672q^3 + 175608q^2 - 20160q + 181440}{8640q^8},\\
b_2 &= -\frac{-204q^8 + 137q^7 + 372q^6 + 259q^5 + 3654q^4 - 6876q^3 + 45792q^2 - 10080q + 45360}{3780q^8},\\
b_3 &= \frac{179q^8 - 1068q^7 + 403q^6 + 2184q^5 + 20286q^4 - 46656q^3 + 195768q^2 - 60480q + 181440}{40320q^8},\\
b_4 &= -\frac{-3q^8 + 53q^7 - 276q^6 + 7q^5 + 4158q^4 - 9036q^3 + 26676q^2 - 10080q + 22680}{22680q^8},\\
b_5 &= \frac{3q^8 - 100q^7 + 1635q^6 - 13160q^5 + 58590q^4 - 106560q^3 + 236088q^2 - 100800q + 181440}{1814400q^8}.
\end{aligned}
$$

When $N = 6$:

$$
\begin{aligned}
a_0 &= \frac{51693q^{10} - 856q^8 + 18721q^6 + 1433070q^4 + 10994040q^2 + 9979200}{129600q^{10}},\\
a_1 &= -\frac{-73191q^{10} + 48926q^9 - 22097q^8 + 10306q^7 + 35192q^6 - 1980q^5 + 2951028q^4 + 1504080q^3 + 22169520q^2 + 1814400q + 19958400}{302400q^{10}},\\
a_2 &= \frac{13053q^{10} + 8834q^9 - 21784q^8 + 23242q^7 + 5185q^6 + 1476q^5 + 1617966q^4 + 1564560q^3 + 11356920q^2 + 1814400q + 9979200}{241920q^{10}},\\
a_3 &= -\frac{-4839q^{10} - 28638q^9 - 6833q^8 + 78966q^7 - 69640q^6 + 64908q^5 + 3811572q^4 + 4996080q^3 + 23621040q^2 + 5443200q + 19958400}{1088640q^{10}},\\
a_4 &= \frac{237q^{10} + 4372q^9 + 24104q^8 + 13892q^7 - 93599q^6 + 160200q^5 + 2414574q^4 + 3612960q^3 + 12445560q^2 + 3628800q + 9979200}{1814400q^{10}},\\
a_5 &= -\frac{-39q^{10} - 770q^9 - 13937q^8 - 111430q^7 - 166408q^6 + 1035540q^5 + 6500340q^4 + 9939600q^3 + 26524080q^2 + 9072000q + 19958400}{19958400q^{10}},\\
a_6 &= \frac{-3q^{10} + 198q^9 + 2024q^8 + 19998q^7 + 239041q^6 + 1324620q^5 + 4548654q^4 + 6629040q^3 + 14259960q^2 + 5443200q + 9979200}{119750400q^{10}},\\
b_1 &= \frac{73191q^{10} + 48926q^9 + 22097q^8 + 10306q^7 - 35192q^6 - 1980q^5 - 2951028q^4 + 1504080q^3 - 22169520q^2 + 1814400q - 19958400}{302400q^{10}},\\
b_2 &= \frac{13053q^{10} - 8834q^9 - 21784q^8 - 23242q^7 + 5185q^6 - 1476q^5 + 1617966q^4 - 1564560q^3 + 11356920q^2 - 1814400q + 9979200}{241920q^{10}},\\
b_3 &= \frac{4839q^{10} - 28638q^9 + 6833q^8 + 78966q^7 + 69640q^6 + 64908q^5 - 3811572q^4 + 4996080q^3 - 23621040q^2 + 5443200q - 19958400}{1088640q^{10}},\\
b_4 &= \frac{237q^{10} - 4372q^9 + 24104q^8 - 13892q^7 - 93599q^6 - 160200q^5 + 2414574q^4 - 3612960q^3 + 12445560q^2 - 3628800q + 9979200}{1814400q^{10}},\\
b_5 &= \frac{39q^{10} - 770q^9 + 13937q^8 - 111430q^7 + 166408q^6 + 1035540q^5 - 6500340q^4 + 9939600q^3 - 26524080q^2 + 9072000q - 19958400}{19958400q^{10}},\\
b_6 &= \frac{-3q^{10} - 198q^9 + 2024q^8 - 19998q^7 + 239041q^6 - 1324620q^5 + 4548654q^4 - 6629040q^3 + 14259960q^2 - 5443200q + 9979200}{119750400q^{10}}.
\end{aligned}
$$