On the existence of nonoscillatory phase functions for second order ordinary differential equations in the high-frequency regime

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Abstract

We observe that solutions of a large class of highly oscillatory second order linear ordinary differential equations can be approximated using nonoscillatory phase functions. In addition, we describe numerical experiments which illustrate several implications of this fact. For example, that many special functions of great interest — such as the Bessel functions J_ν and Y_ν — can be evaluated accurately using a number of operations which is $O(1)$ in the order ν. The present paper is devoted to the development of an analytical apparatus. Numerical aspects of this work will be reported at a later date.

Keywords: Special functions, ordinary differential equations, phase functions

1. Introduction

Given a differential equation

$$y''(t) + \lambda^2 q(t) y(t) = 0 \quad \text{for all } 0 \leqslant t \leqslant 1, \tag{1}$$

where λ is a real number and $q : [0,1] \to \mathbb{R}$ is smooth and strictly positive, a sufficiently smooth $\alpha : [0,1] \to \mathbb{R}$ is a phase function for (1) if the pair of functions u, v defined by the formulas

$$u(t) = \frac{\cos(\alpha(t))}{|\alpha'(t)|^{1/2}} \tag{2}$$

and

$$v(t) = \frac{\sin(\alpha(t))}{|\alpha'(t)|^{1/2}} \tag{3}$$

form a basis in the space of solutions of (1). Phase functions have been extensively studied: they were first introduced in [9], play a key role in the theory of global transformations of ordinary differential equations [3, 10], and are an important element in the theory of special functions [16, 6, 11, 1].

Despite this long history, a useful property of phase functions appears to have been overlooked. Specifically, that when the function q is nonoscillatory, solutions of the equation (1) can be accurately represented using a nonoscillatory phase function.

This is somewhat surprising since α is a phase function for (1) if and only if it satisfies the third order

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nonlinear ordinary differential equation

$$\left(\alpha'(t)\right)^2 = \lambda^2 q(t) - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 \quad \text{for all} \quad 0 \leqslant t \leqslant 1. \tag{4}$$

The equation (4) was introduced in [9], and and we will refer to it as Kummer's equation. The form of (4) and the appearance of λ in it suggests that its solutions will be oscillatory — and most of them are. However, Bessel's equation

$$y''(t) + \left(1 - \frac{\lambda^2 - 1/4}{t^2}\right) y(t) = 0 \quad \text{for all} \quad 0 < t < \infty \tag{5}$$

furnishes a nontrivial example of an equation which admits a nonoscillatory phase function regardless of the value of λ. If we define u, v by the formulas

$$u(t) = \sqrt{\frac{\pi t}{2}} J_\lambda(t) \tag{6}$$

and

$$v(t) = \sqrt{\frac{\pi t}{2}} Y_\lambda(t), \tag{7}$$

where J_λ and Y_λ denote the Bessel functions of the first and second kinds of order λ, and let α be defined by the relations (2),(3), then

$$\alpha'(t) = \frac{2}{\pi t}\frac{1}{J_\lambda^2(t) + Y_\lambda^2(t)}. \tag{8}$$

It can be easily verified that (8) is a nonoscillatory. The existence of this nonoscillatory phase function for Bessel's equation is the basis of several methods for the evaluation of Bessel functions of large orders and for the computation of their zeros [6, 8, 15].

The general situation is not quite so favorable: there need not exist a nonoscillatory function α such that (2) and (3) are exact solutions of (1). However, assuming that q is nonoscillatory and λ is sufficiently large, there exists a nonoscillatory function α such that (2), (3) approximate solutions of (1) with spectral accuracy (i.e., the approximation errors decay exponentially with λ).

To see that this claim is plausible, we apply Newton's method for the solution of nonlinear equations to Kummer's equation (4). In doing so, it will be convenient to move the setting of our analysis from the interval $[0, 1]$ to the real line so that we can use the Fourier transform to quantity the notion of "nonoscillatory." Suppose that the extension of q to the real line is smooth and strictly positive, and such that $\log(q)$ is a smooth function with rapidly decaying Fourier transform. Letting

$$\left(\alpha'(t)\right)^2 = \lambda^2 \exp(r(t)) \tag{9}$$

in (4) yields the logarithm form of Kummer's equation:

$$r''(t) - \frac{1}{4}\left(r'(t)\right)^2 + 4\lambda^2 \left(\exp(r(t)) - q(t)\right) = 0 \quad \text{for all} \quad t \in \mathbb{R}. \tag{10}$$

We use $\{r_n\}$ to denote the sequence of Newton iterates for the equation (10) obtained from the initial guess

$$r_0(t) = \log(q(t)). \tag{11}$$

The function r_0 corresponds to the first order WKB approximations for (1). That is to say that if we insert the associated phase function

$$\alpha_0(t) = \lambda \int_0^t \exp\left(\frac{1}{2} r_0(u)\right) \, du = \lambda \int_0^t \sqrt{q(u)} du \tag{12}$$

into (2),(3), then

$$u(t) = q^{-1/4}(t)\cos\left(\lambda \int_0^t \sqrt{q(u)}\,du\right) \tag{13}$$

and

$$v(t) = q^{-1/4}(t)\sin\left(\lambda \int_0^t \sqrt{q(u)}\,du\right). \tag{14}$$

For each $n \geqslant 0$, r_{n+1} is obtained from r_n by solving the linearized equation

$$h''(t) - \frac{1}{2}r_n'(t)h'(t) + 4\lambda^2 \exp\left(r_n(t)\right)h(t) = f_n(t) \quad \text{for all } \ t \in \mathbb{R}, \tag{15}$$

where

$$f_n(t) = -r_n''(t) + \frac{1}{4}\left(r_n'(t)\right)^2 - 4\lambda^2\left(\exp\left(r_n(t)\right) - q(t)\right), \tag{16}$$

and letting

$$r_{n+1}(t) = r_n(t) + h(t). \tag{17}$$

By introducing the change of variables

$$x(t) = \int_0^t \exp\left(\frac{r_n(u)}{2}\right)\,du \tag{18}$$

into (15), we transform it into the inhomogeneous Helmholtz equation

$$h''(x) + 4\lambda^2 h(x) = g_n(x) \quad \text{for all } \ x \in \mathbb{R}, \tag{19}$$

where

$$g_n(x) = \exp\left(-r_n(x)\right) f_n(x). \tag{20}$$

Suppose that $\widehat{g_n}$ decays rapidly (when $n = 0$, this is a consequence of our assumption that $\log(q)$ has a rapidly decaying Fourier transform) and let h^* be the solution of (19) whose Fourier transform is

$$\widehat{h^*}(\xi) = \frac{\widehat{g_n}(\xi)}{4\lambda^2 - \xi^2}. \tag{21}$$

Since $\widehat{h^*}(\xi)$ is singular when $\xi = \pm 2\lambda$, h^* will necessarily have a component which oscillates at frequency 2λ. However, according to (21), the $L^\infty(\mathbb{R})$ norm of that component is

$$\frac{\widehat{g_n}(2\lambda)}{4\lambda}. \tag{22}$$

In fact, by rearranging (21) as

$$\widehat{h^*}(\xi) = \frac{1}{4\lambda}\left(\frac{\widehat{g_n}(\xi)}{2\lambda - \xi} + \frac{\widehat{g_n}(\xi)}{2\lambda + \xi}\right) \tag{23}$$

and decomposing each of the terms on the right-hand side of (23) as

$$\frac{\widehat{g_n}(\xi)}{2\lambda \pm \xi} = \frac{1}{4\lambda}\left(\frac{\widehat{g_n}(\xi) - \widehat{g_n}(\mp 2\lambda)\exp\left(-(2\lambda \pm \xi)^2\right)}{2\lambda \pm \xi} + \widehat{g_n}(\mp 2\lambda)\frac{\exp\left(-(2\lambda \pm \xi)^2\right)}{2\lambda \pm \xi}\right), \tag{24}$$

we obtain

$$h^*(x) = h_0(x) + h_1(x), \tag{25}$$

where h_0 is defined by the formula

$$\widehat{h_0}(\xi) = \frac{1}{4\lambda}\left(\frac{\widehat{g_n}(\xi) - \widehat{g_n}(-2\lambda)\exp\left(-(2\lambda + \xi)^2\right)}{2\lambda + \xi} + \frac{\widehat{g_n}(\xi) - \widehat{g_n}(2\lambda)\exp\left(-(2\lambda - \xi)^2\right)}{2\lambda - \xi}\right), \tag{26}$$

and h_1 is defined by the formula

$$\widehat{h_1}(\xi) = \frac{1}{4\lambda}\left(\widehat{g_n}(-2\lambda)\frac{\exp\left(-(2\lambda+\xi)^2\right)}{2\lambda+\xi} + \widehat{g_n}(2\lambda)\frac{\exp\left(-(2\lambda-\xi)^2\right)}{2\lambda-\xi}\right). \tag{27}$$

Since the factor in the denominator in (26) has been canceled and both $\widehat{g_n}$ and the Gaussian function are smooth and rapidly decaying, $\widehat{h_0}$ is also smooth and rapidly decaying. Meanwhile, a straightforward calculation shows that the Fourier transform of

$$\frac{1}{2i}\operatorname{erf}\left(\frac{x}{2}\right)\exp(2\lambda i x) \tag{28}$$

is

$$\frac{\exp\left(-(2\lambda-\xi)^2\right)}{2\lambda-\xi}, \tag{29}$$

so that (27) implies that

$$h_1(x) = \frac{1}{4\lambda}\left(\widehat{g_n}(-2\lambda)\frac{1}{2i}\operatorname{erf}\left(\frac{x}{2}\right)\exp(2\lambda i x) - \widehat{g_n}(2\lambda)\frac{1}{2i}\operatorname{erf}\left(\frac{x}{2}\right)\exp(-2\lambda i x)\right). \tag{30}$$

Since g_n is real-valued, $\widehat{g_n}(2\lambda) = \widehat{g_n}(-2\lambda)$. Inserting this into (30) yields

$$h_1(x) = \frac{\widehat{g}(2\lambda)}{4\lambda}\operatorname{erf}\left(\frac{x}{2}\right)\sin\left(2\lambda x\right), \tag{31}$$

which makes it clear that the $L^\infty(\mathbb{R})$ norm of h_1 is $(4\lambda)^{-1}\widehat{g_n}(2\lambda)$.

In (25), the solution of (19) is decomposed as the sum of a nonoscillatory function h_0 and a highly oscillatory function h_1 of small magnitude. However, the solution of (15) is actually given by the function

$$h^*(x(t)) = h_0(x(t)) + h_1(x(t)) \tag{32}$$

obtained by reversing the change of variables (18). But since $x(t)$ is nonoscillatory and the Fourier transform of $h_0(x)$ decays rapidly, we expect that the composition $h_0(x(t))$ will also have a rapidly decaying Fourier transform. The $L^\infty(\mathbb{R})$ norm of $h_1(x(t))$ is, of course, the same as that of $h_1(x)$. So the solution of the linearized equation (15) can be written as the sum of a nonoscillatory function $h_0(x(t))$ and a highly oscillatory function $h_1(x(t))$ of negligible magnitude.

If, in each iteration of the Newton procedure, we approximate the solution of (15) by constructing $h^*(x(t))$ and discarding the oscillatory term $h_1(t(x))$ of small magnitude, then it is plausible that we will arrive at an approximate solution $r(t)$ of the logarithm form of Kummer's equation which is nonoscillatory, assuming the Fourier transform of $r_0(t) = \log(q(t))$ decays rapidly enough and λ is sufficiently large.

Most of the remainder of this paper is devoted to developing a rigorous argument to replace the preceding heuristic discussion. In Section 2, we summarize a number of well-known mathematical facts to be used throughout this article. In Section 3, we reformulate Kummer's equation as a nonlinear integral equation. Once that is accomplished, we are in a position to state the principal result of the paper and discuss its implications; this is done in Section 4. The proof of this principal result is contained in Sections 5, 6, 7 and 8.

In Section 9, we present the results of numerical experiments concerning the evaluation of special functions. The details of our numerical algorithm will be reported at a later date.

We conclude with a few brief remarks in Section 10.

2. Preliminaries

2.1. Schwartz functions and tempered distributions

We say that $\varphi \in C^{\infty}(\mathbb{R})$ is is a Schwartz function if φ and all of its derivatives decay faster than any polynomial. That is, if

$$\sup_{t\in\mathbb{R}} |t^i \varphi^{(j)}(t)| < \infty \tag{33}$$

for all pairs i, j of nonnegative integers. The set of all Schwartz functions is denoted by $S(\mathbb{R})$. It is endowed with the topology generated by the family of seminorms

$$\|\varphi\|_k = \sum_{j=0}^{k} \sup_{t\in\mathbb{R}} \left| t^k \varphi^{(j)}(x) \right| \quad k = 0, 1, 2, \ldots, \tag{34}$$

so that a sequence $\{\varphi_n\}$ of functions in $S(\mathbb{R})$ converges to φ in $S(\mathbb{R})$ if and only if

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_k = 0 \quad \text{for all} \quad k = 0, 1, 2, \ldots. \tag{35}$$

We denote the space of continuous linear functionals on $S(\mathbb{R})$, which are known as tempered distributions, by $S'(\mathbb{R})$.

2.2. Convention for the Fourier transform

We define the Fourier transform of a function $f \in L^1(\mathbb{R})$ via the formula

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} \exp(-it\xi) f(t)\, dt. \tag{36}$$

so that

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi), \tag{37}$$

$$\widehat{f \cdot g}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi - \eta)\widehat{g}(\eta)\, d\eta, \tag{38}$$

and

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(it\xi) f(t)\, dt. \tag{39}$$

2.3. Modified Bessel functions

The modified Bessel function $K_\nu(t)$ of the first kind of order ν is defined for $t \in \mathbb{R}$ and $\nu \in \mathbb{C}$ by the formula

$$K_\nu(t) = \int_0^{\infty} \exp\left(-t\cosh(t)\right)\cosh(\nu t)\, dt. \tag{40}$$

The following bound on the ratio of $K_{\nu+1}$ to K_ν can be found in [14].

Theorem 1. *Suppose that $t > 0$ and $\nu > 0$ are real numbers. Then*

$$\frac{K_{\nu+1}(t)}{K_\nu(t)} < \frac{\nu + \sqrt{\nu^2 + t^2}}{t} \leqslant \frac{2\nu}{t} + 1. \tag{41}$$

2.4. The binomial theorem

A proof of the following can be found in [13], as well as many other sources.

Theorem 2. *Suppose that r is a real number, and that y is a real number such that $|y| < 1$. Then*

$$(1+y)^r = \sum_{k=0}^{\infty} \frac{\Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k+1)} y^k. \tag{42}$$

2.5. Fréchet derivatives and the contraction mapping principle

Given Banach spaces X, Y and a mapping $f : X \to Y$ between them, we say that f is Fréchet differentiable at $x \in X$ if there exists a bounded linear operator $X \to Y$, denoted by f'_x, such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - f'_x[h]\|}{\|h\|} = 0. \tag{43}$$

Theorem 3. *Suppose that X and Y are a Banach spaces and that $f : X \to Y$ is Fréchet differentiable at every point of X. Suppose also that D is a convex subset of X, and that there exists a real number $M > 0$ such that*

$$\|f'_x\| \leqslant M \tag{44}$$

for all $x \in D$. Then

$$\|f(x) - f(y)\| \leqslant M\|x - y\| \tag{45}$$

for all x and y in D.

Suppose that $f : X \to X$ is a mapping of the Banach space X into itself. We say that f is contractive on a subset D of X if there exists a real number $0 < \alpha < 1$ such that

$$\|f(x) - f(y)\| \leqslant \alpha\|x - y\| \tag{46}$$

for all $x, y \in D$. Moreover, we say that $\{x_n\}_{n=0}^{\infty}$ is a sequence of fixed point iterates for f if $x_{n+1} = f(x_n)$ for all $n \geqslant 0$.

Theorem 3 is often used to show that a mapping is contractive so that the following result can be applied.

Theorem 4. *(The Contraction Mapping Principle) Suppose that D is a closed subset of a Banach space X. Suppose also that $f : X \to X$ is contractive on D and $f(D) \subset D$. Then the equation*

$$x = f(x) \tag{47}$$

has a unique solution $\sigma^* \in D$. Moreover, any sequence of fixed point iterates for the function f which contains an element in D converges to σ^*.

A discussion of Fréchet derivatives and proofs of Theorems 3 and 4 can be found, for instance, in [18].

2.6. Schwarzian derivatives

The Schwarzian derivative of a smooth function $f : \mathbb{R} \to \mathbb{R}$ is

$$\{f, t\} = \frac{f'''(t)}{f'(t)} - \frac{3}{2}\left(\frac{f''(t)}{f'(t)}\right)^2. \tag{48}$$

If the function $x(t)$ is a diffeomorphism of the real line (that is, a smooth, invertible mapping $\mathbb{R} \to \mathbb{R}$), then the Schwarzian derivative of $x(t)$ can be related to the Schwarzian derivative of its inverse $t(x)$;

in particular,

$$\{x,t\} = -\left(\frac{dx}{dt}\right)^2 \{t,x\}. \tag{49}$$

The identity (49) can be found, for instance, in Section 1.13 of [11].

3. Integral equation formulation

In this section, we reformulate Kummer's equation

$$\left(\alpha'(t)\right)^2 = \lambda^2 q(t) - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 \tag{50}$$

as a nonlinear integral equation. As in the introduction, we assume that the function q has been extended to the real line and we seek a function α which satisfies (50) on the real line.

By letting

$$\left(\alpha'(t)\right)^2 = \lambda^2 \exp(r(t)) \tag{51}$$

in (50), we obtain the equation

$$r''(t) - \frac{1}{4}\left(r'(t)\right)^2 + 4\lambda^2\left(\exp(r(t)) - q(t)\right) = 0 \quad \text{for all} \quad t \in \mathbb{R}. \tag{52}$$

We next take r to be of the form

$$r(t) = \log(q(t)) + \delta(t), \tag{53}$$

which results in

$$\delta''(t) - \frac{1}{2}\frac{q'(t)}{q(t)}\delta'(t) - \frac{1}{4}\left(\delta'(t)\right)^2 + 4\lambda^2 q(t)\left(\exp(\delta(t)) - 1\right) = q(t)p(t), \quad \text{for all} \quad t \in \mathbb{R}, \tag{54}$$

where p is defined by the formula

$$p(t) = \frac{1}{q(t)}\left(\frac{5}{4}\left(\frac{q'(t)}{q(t)}\right)^2 - \frac{q''(t)}{q(t)}\right). \tag{55}$$

Note that the function p appears in the standard error analysis of WKB approximations (see, for instance, [12]). Expanding the exponential in a power series and rearranging terms yields the equation

$$\delta''(t) - \frac{1}{2}\frac{q'(t)}{q(t)}\delta'(t) + 4\lambda^2 q(t)\delta(t) - \frac{1}{4}\left(\delta'(t)\right)^2 + 4\lambda^2 q(t)\left(\frac{(\delta(t))^2}{2} + \frac{(\delta(t))^3}{3!} + \cdots\right) = q(t)p(t). \tag{56}$$

Applying the change of variables

$$x(t) = \int_0^t \sqrt{q(u)}\, du \tag{57}$$

transforms (56) into

$$\delta''(x) + 4\lambda^2\delta(x) - \frac{1}{4}\left(\delta'(x)\right)^2 + 4\lambda^2\left(\frac{(\delta(x))^2}{2} + \frac{(\delta(x))^3}{3!} + \cdots\right) = p(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{58}$$

At first glance, the relationship between the function $p(x)$ appearing in (58) and the coefficient $q(t)$ in the ordinary differential equation (1) is complex. However, the function $p(t)$ defined via (55) is related to the Schwarzian derivative (see Section 2.6) of the function $x(t)$ defined in (57) via the formula

$$p(t) = -\frac{2}{q(t)}\{x,t\} = -2\left(\frac{dt}{dx}\right)^2\{x,t\}. \tag{59}$$

It follows from (59) and Formula (49) in Section 2.6 that

$$p(x) = 2\{t, x\}. \tag{60}$$

That is to say: p, when viewed as a function of x, is simply twice the Schwarzian derivative of t with respect to x.

It is also notable that the part of (58) which is linear in δ is a constant coefficient Helmholtz equation. This suggests that we form an integral equation for (58) using a Green's function for the Helmholtz equation. To that end, we define the integral operator T for functions $f \in L^1(\mathbb{R})$ via the formula

$$T[f](x) = \frac{1}{4\lambda}\int_{-\infty}^{\infty} \sin(2\lambda|x-y|)\, f(y)\, dy \tag{61}$$

The following theorem summarizes the relevant properties of the operator T.

Theorem 5. *Suppose that $\lambda > 0$ is a real number, and that the operator T is defined as in (61). Suppose also that $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$. Then:*

1. *$T[f](x)$ is an element of $C^2(\mathbb{R})$;*
2. *$T[f](x)$ is a solution of the ordinary differential equation*
$$y''(x) + 4\lambda^2 y(x) = f(x) \quad \textit{for all } x \in \mathbb{R}; \textit{ and}$$
3. *the Fourier transform of $T[f](x)$ is the principal value of*
$$\frac{\widehat{f}(\xi)}{4\lambda^2-\xi^2} = \frac{1}{4\lambda}\left(\frac{\widehat{f}(\xi)}{2\lambda-\xi} + \frac{\widehat{f}(\xi)}{2\lambda+\xi}\right).$$

Proof. We observe that

$$\begin{aligned} T[f](x) =& \frac{1}{4\lambda}\int_{-\infty}^{x} \sin(2\lambda(x-y))\, f(y)\, dy + \frac{1}{4\lambda}\int_{x}^{\infty} \sin(2\lambda(y-x))\, f(y)\, dy \\ =& \frac{1}{4\lambda}\sin(2\lambda x)\int_{-\infty}^{x} \cos(2\lambda y) f(y)\, dy - \frac{1}{4\lambda}\cos(2\lambda x)\int_{-\infty}^{x} \sin(2\lambda y) f(y)\, dy \\ &+ \frac{1}{4\lambda}\cos(2\lambda x)\int_{x}^{\infty} \sin(2\lambda y) f(y)\, dy - \frac{1}{4\lambda}\sin(2\lambda x)\int_{x}^{\infty} \cos(2\lambda y) f(y)\, dy \end{aligned} \tag{62}$$

for all $x \in \mathbb{R}$. We differentiate both sides of (62) with respect to x, apply the Lebesgue dominated convergence theorem to each integral (this is permissible since the sine and cosine functions are bounded and $f \in L^1(\mathbb{R})$) and combine terms in order to conclude that $T[f]$ is differentiable everywhere and

$$\frac{d}{dx} T[f](x) = = \frac{1}{2}\int_{-\infty}^{\infty} \cos(2\lambda|x-y|)\operatorname{sign}(x-y) f(y)\, dy \tag{63}$$

for all $x \in \mathbb{R}$. In the same fashion, we conclude that

$$\left(\frac{d}{dx}\right)^2 T[f](x) = f(x) - \lambda\int_{-\infty}^{\infty} \sin(2\lambda|x-y|)\, f(y)\, dy \tag{64}$$

for all $x \in \mathbb{R}$. Since f is continuous by assumption and the second term appearing on the right-hand side in (64) is a continuous function of x by the Lebesgue dominated convergence theorem, we see from (64) that $T[f]$ is twice continuously differentiable. By combining (64) and (61), we conclude that $T[f]$ is a solution of the ordinary differential equation

$$y''(x) + 4\lambda^2 y(x) = f(x) \quad \text{for all } x \in \mathbb{R}. \tag{65}$$

We now define the function g through the formula

$$\widehat{g}(\xi) = \frac{1}{4\lambda}\left(\frac{1}{2\lambda-\xi} + \frac{1}{2\lambda+\xi}\right). \tag{66}$$

It is well known that the Fourier transform of the principal value of $1/x$ is the function

$$-i\pi\,\mathrm{sign}(x); \tag{67}$$

see, for instance, [17] or [7]. It follows readily that the inverse Fourier transform of the principal value of

$$\frac{1}{2\lambda \pm \xi} \tag{68}$$

is

$$\pm\frac{1}{2i}\exp\left(\mp 2\lambda i x\right)\mathrm{sign}(x). \tag{69}$$

From this and (66), we conclude that

$$\begin{aligned} g(x) &= \frac{1}{4\lambda}\left(\frac{1}{2i}\exp\left(-2\lambda i x\right)\mathrm{sign}(x) - \frac{1}{2i}\exp\left(2\lambda i x\right)\mathrm{sign}(x)\right) \\ &= \frac{1}{4\lambda}\sin\left(2\lambda|x|\right). \end{aligned} \tag{70}$$

In particular, $T\left[\,f\,\right]$ is the convolution of f with g. As a consequence,

$$\widehat{T\left[\,f\,\right]}(\xi) = \widehat{g}(\xi)\widehat{f}(\xi) = \frac{1}{4\lambda}\left(\frac{\widehat{f}(\xi)}{2\lambda-\xi} + \frac{\widehat{f}(\xi)}{2\lambda+\xi}\right), \tag{71}$$

which is the third and final conclusion of the theorem. □

In light of Theorem 5, it is clear that introducing the representation

$$\delta(x) = T\left[\,\sigma\,\right](x) \tag{72}$$

into (58) yields the nonlinear integral equation

$$\sigma(x) = S\left[T\left[\,\sigma\,\right]\right](x) + p(x) \quad \text{for all } \ x \in \mathbb{R}, \tag{73}$$

where S is the operator defined for functions $f \in C^1\left(\mathbb{R}\right)$ by the formula

$$S\left[\,f\,\right](x) = \frac{\left(f'(x)\right)^2}{4} - 4\lambda^2\left(\frac{\left(f(x)\right)^2}{2!} + \frac{\left(f(x)\right)^3}{3!} + \frac{\left(f(x)\right)^4}{4!} + \cdots\right). \tag{74}$$

The following theorem is immediately apparent from the procedure used to transform Kummer's equation (50) into the nonlinear integral equation (73).

Theorem 6. *Suppose that $\lambda > 0$ is a real number, that $q : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable, strictly positive function, that $x(t)$ is defined by (57), and that $p(x)$ is defined via (60). Suppose also that $\sigma \in L^1\left(\mathbb{R}\right) \cap C\left(\mathbb{R}\right)$ is a solution of the integral equation (73), that δ is defined via the formula*

$$\delta(x) = T\left[\,\sigma\,\right](x) = \frac{1}{4\lambda}\int_{-\infty}^{\infty}\sin\left(2\lambda\left|x-y\right|\right)\sigma(y)\,dy, \tag{75}$$

and that the function α is defined by the formula

$$\alpha(t) = \lambda\int_0^t \sqrt{q(u)}\exp\left(\frac{\delta(x(u))}{2}\right)\,du. \tag{76}$$

Then:

1. *$\delta(x)$ is a twice continuously differentiable solution of (58);*
2. *$\delta(x(t))$ is a twice continuously differentiable solution of of (56);*
3. *$\alpha(t)$ is three times continuously differentiable solution of (50); and*
4. *$\alpha(t)$ is a phase function for the ordinary differential equation*

$$y''(t) + \lambda^2 q(t) y(t) = 0 \quad \textit{for all} \quad 0 \leqslant t \leqslant 1. \tag{77}$$

In the event that $f \in L^2(\mathbb{R})$, the integral

$$\frac{1}{2\lambda} \int_{-\infty}^{\infty} \sin\left(2\lambda\left|x-y\right|\right)\, f(y)\, dy \tag{78}$$

defining $T[f]$ need not be absolutely integrable. However, if the Fourier transform of f is compactly supported on the interval $(-2\lambda, 2\lambda)$ then

$$\frac{\widehat{f}(\xi)}{4\lambda^2 - \xi^2}, \tag{79}$$

which is formally the Fourier transform of $T[f]$, is a $L^1(\mathbb{R})$ function. In this event, we define $T[f]$ via the inverse Fourier transform; that is,

$$T[f](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ix\xi)\, \frac{\widehat{f}(\xi)}{4\lambda^2 - \xi^2} d\xi. \tag{80}$$

The absolute convergence of this integral is a consequence of the assumption that $\widehat{f}$ is compactly supported in $(-2\lambda, 2\lambda)$. Since $T[f]$ and f are inverse Fourier transforms of compactly supported functions, both are entire functions (in particular, they are both twice continuously differentiable). Moreover, we observe that the Fourier transform of the function

$$T[f]''(x) + 4\lambda^2 T[f](x) \tag{81}$$

is

$$-\xi^2 \widehat{T[f]} + 4\lambda^2 \widehat{T[f]} = -\xi^2 \frac{\widehat{f}(\xi)}{4\lambda^2 - \xi^2} + 4\lambda^2 \frac{\widehat{f}(\xi)}{4\lambda^2 - \xi^2} = \widehat{f}(\xi), \tag{82}$$

from which we conclude that

$$T[f]''(x) + 4\lambda T[f](x) = f(x) \tag{83}$$

almost everywhere. The continuity of f and $T[f]$ now imply that (83) in fact holds everywhere; that is, $T[f]$ is a solution of the inhomogeneous Helmholtz equation

$$y''(x) + 4\lambda^2 y(x) = f(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{84}$$

4. Overview and statement of the principal result

The nonlinear integral equation (73) is not solvable for arbitrary p. In this article, we will show that when the function p is nonoscillatory, there exists a nonoscillatory function σ and a function ν of magnitude which decays exponentially in λ such that

$$\sigma(x) = S[T[\sigma]](x) + p(x) + \nu(x). \quad \text{for all} \quad x \in \mathbb{R} \tag{85}$$

The next theorem, which is the principal result of this article, makes these statements precise. Its proof is given in Sections 5, 6, 7 and 8.

Theorem 7. *Suppose that $q \in C^\infty(\mathbb{R})$ is a strictly positive, that $x(t)$ is defined by the formula*

$$x(t) = \int_0^t \sqrt{q(u)}\, du, \tag{86}$$

and that the function p defined via the formula

$$p(x) = 2\{t, x\} \tag{87}$$

is an element of $S(\mathbb{R})$. Suppose furthermore that there exist positive real numbers λ, Γ and a such that

$$\lambda \geqslant \max\left\{\frac{2}{a}, 2\Gamma, \sqrt{\frac{2\Gamma}{\pi}}\right\} \tag{88}$$

and

$$|\widehat{p}(\xi)| \leqslant \Gamma \exp\left(-a\,|\xi|\right) \quad \textit{for all} \;\; \xi \in \mathbb{R}. \tag{89}$$

Then there exists a function $\sigma \in S(\mathbb{R})$ and an infinitely differentiable function ν such that σ is a solution of the nonlinear integral equation

$$\sigma(x) = S\left[T\left[\sigma\right]\right](x) + p(x) + \nu(x), \quad \textit{for all} \;\; x \in \mathbb{R}, \tag{90}$$

$$|\widehat{\sigma}(\xi)| < 2\Gamma \exp\left(-a|\xi|\right) \quad \textit{for all} \;\; |\xi| \leqslant \sqrt{2}\lambda, \tag{91}$$

$$\widehat{\sigma}(\xi) = 0 \quad \textit{for all} \;\; |\xi| > \sqrt{2}\lambda, \tag{92}$$

and

$$\|\nu\|_\infty \leqslant \frac{12\Gamma}{a} \exp\left(-a\lambda\right). \tag{93}$$

Suppose that σ and ν are the functions obtained by invoking Theorem 7. We define the function δ by the formula

$$\delta(t) = T\left[\sigma\right](x(t)), \tag{94}$$

where $x(t)$ is as in (86), the function r by the formula

$$r(t) = \log(q(t)) + \delta(t) \tag{95}$$

and the function α by the formula

$$\alpha(t) = \lambda \int_0^t \sqrt{q(u)}\, \exp\left(\frac{r(u)}{2}\right)\, du. \tag{96}$$

The function α is not a phase function for the original differential equation (1). Suppose, however, that $\tilde{q}$ is a solution of the ordinary differential equation

$$\frac{1}{\tilde{q}(t)}\left(\frac{5}{4}\left(\frac{\tilde{q}'(t)}{\tilde{q}(t)}\right)^2 - \frac{\tilde{q}''(t)}{\tilde{q}(t)}\right) = p(t) + \nu(t) \quad \text{for all} \;\; 0 \leqslant t \leqslant 1. \tag{97}$$

Then the function α defined via (96) is a phase function for the modified second order linear ordinary differential equation

$$y''(t) + \lambda^2 \tilde{q}(t) y(t) = 0 \quad \text{for all} \;\; 0 \leqslant t \leqslant 1. \tag{98}$$

since σ is a solution of (90). Moreover, since the magnitude of ν is small, we expect the difference between solutions of (98) and (1). In particular, the functions u, v defined via the formulas

$$u(t) = \frac{\cos(\alpha(t))}{\sqrt{\alpha'(t)}} \tag{99}$$

$$v(t) = \frac{\sin(\alpha(t))}{\sqrt{\alpha'(t)}}, \tag{100}$$

which are solutions of (98), will closely agree with solutions of (1). A more precise version of the following statement to this effect could be formulated easily using standard results found in textbooks on ordinary differential equations (for instance, [4]).

Theorem 8. *Suppose that the hypotheses of Theorem 7 are satisfied. Suppose further that u and v are the functions defined by the formulas (99) and (100). If λ is sufficiently large, then there exists a constant C which depends on q but not λ, and a basis $\{\tilde{u}, \tilde{v}\}$ in the space of solutions of the second order differential (1) such that*

$$|u(t) - \tilde{u}(t)| \leqslant C \exp(-a\lambda) \tag{101}$$

and

$$|v(t) - \tilde{v}(t)| \leqslant C \exp(-a\lambda) \tag{102}$$

for all $0 \leqslant t \leqslant 1$.

The proof of Theorem 7 is divided amongst Sections 5, 6, 7 and 8. The principal difficulty lies in constructing a function ν such that (85) admits a solution. We accomplish this by introducing a modified integral equation

$$\sigma_b(x) = S\left[T_b\left[\sigma_b\right]\right](x) + p(x), \tag{103}$$

where T_b is a "band-limited" version of T. That is, $T_b\left[f\right]$ is defined via the formula

$$\widehat{T_b\left[f\right]}(\xi) = \widehat{T\left[f\right]}(\xi) b(\xi), \tag{104}$$

where $b(\xi)$ is a $C_c^\infty(\mathbb{R})$ bump function. This modified integral equation is introduced in Section 5. In Section 6, we show that under mild conditions on p and λ, (103) admits a solution σ_b. The argument proceeds by applying the Fourier transform to (103) and using the contraction mapping principle to show that the resulting equation admits a solution. In Section 7, we show that the Fourier transform of the solution σ_b of (103) is exponentially decaying whenever $\widehat{p}$ is exponentially decaying. In Section 8 we use the solution σ_b of (103) in order to construct functions σ and ν which satisfy (85). Moreover, we show that σ can be taken to be an element of the space $S(\mathbb{R})$ of rapidly decaying Schwartz functions (see Section 2.1), that the Fourier transform of σ decays exponentially with λ, that ν is infinitely differentiable, and that the $L^\infty(\mathbb{R})$ norm of ν has $L^\infty(\mathbb{R})$ also decays exponentially with λ.

5. Band-limited integral equation

In this section, we introduce a "band-limited" version of the operator T and use it to form an alternative to the integral equation (73).

Let $b(\xi)$ be any infinitely differentiable function such that

1. $b(\xi) = 1$ for all $|\xi| \leqslant \lambda$,
2. $0 \leqslant b(\xi) \leqslant 1$ for all $\lambda \leqslant |\xi| \leqslant \sqrt{2}\lambda$, and
3. there exists an $\epsilon > 0$ such that $b(\xi) = 0$ for all $|\xi| > \sqrt{2}\lambda - \epsilon$.

We define $T_b\left[f\right]$ for functions $f \in L^1(\mathbb{R})$ via the formula

$$\widehat{T_b\left[f\right]}(\xi) = \widehat{f}(\xi) \frac{b(\xi)}{4\lambda^2 - \xi^2} \tag{105}$$

We will refer to T_b as the band-limited version of the operator T and and we call the nonlinear integral equation

$$\sigma_b(x) = S\left[T_b\left[\sigma_b\right]\right](x) + p(x) \quad \text{for all } \ x \in \mathbb{R} \tag{106}$$

obtained by replacing T with T_b in (73) the "band-limited" version of (73).

Since T_b is a Fourier multiplier, it is convenient to analyze (106) in the Fourier domain rather than the space domain. We now introduce notation which will allow us to write down the equation obtained by applying the Fourier transform to both sides of (106).

We let W_b and $\widetilde{W}_b$ be the linear operators defined for $f \in L^1(\mathbb{R})$ via the formulas

$$W_b\left[f\right](\xi) = f(\xi)\frac{b(\xi)}{4\lambda^2 - \xi^2} \tag{107}$$

and

$$\widetilde{W}_b\left[f\right](\xi) = f(\xi)\frac{b(\xi)i\xi}{4\lambda^2 - \xi^2}, \tag{108}$$

where $b(\xi)$ is the function used to define the operator T_b.

For functions $f \in L^1(\mathbb{R})$, it is standard to denote the Fourier transform of the function $\exp(f(x))$ by $\exp^*[f]$; that is,

$$\exp^*[f](\xi) = 2\pi\delta(\xi) + f(\xi) + \frac{f * f(\xi)}{2!(2\pi)} + \frac{f * f * f(\xi)}{3!(2\pi)^2} + \cdots. \tag{109}$$

In (109), δ defers to the delta distribution and $f * f * \cdots * f$ denotes repeated convolution of the function f with itself. The Fourier transform of $\exp(f(x))$ never appears in this paper; however, we will encounter the Fourier transforms of the functions

$$\exp(f(x)) - 1 \tag{110}$$

and

$$\exp(f(x)) - f(x) - 1. \tag{111}$$

So, in analogy with the definition (109), we define $\exp_1^*[f]$ for $f \in L^1(\mathbb{R})$ by the formula

$$\exp_1^*[f](\xi) = f(\xi) + \frac{f * f(\xi)}{2!(2\pi)} + \frac{f * f * f(\xi)}{3!(2\pi)^2} + \cdots, \tag{112}$$

and we define $\exp_2^*[f]$ for $f \in L^1(\mathbb{R})$ via the formula

$$\exp_2^*[f](\xi) = \frac{f * f(\xi)}{2!(2\pi)} + \frac{f * f * f(\xi)}{3!(2\pi)^2} + \cdots. \tag{113}$$

That is, $\exp_1^*[f]$ is obtained by truncating the leading term of $\exp^*[f]$ and $\exp_2^*[f]$ is obtained by truncated the first two leading terms of $\exp^*[f]$.

Finally, we define functions $\psi(\xi)$ and $v(\xi)$ using the formulas

$$\psi(\xi) = \widehat{\sigma}_b(\xi) \tag{114}$$

and

$$v(\xi) = \widehat{p}(\xi). \tag{115}$$

Applying the Fourier transform to both sides of (106) results in the nonlinear equation

$$\psi(\xi) = R\left[\psi\right](\xi), \tag{116}$$

where $R[f]$ is defined for $f \in L^1(\mathbb{R})$ by the formula

$$R[f](\xi) = \frac{1}{8\pi}\widetilde{W}_b[f] * \widetilde{W}_b[f](\xi) - 4\lambda^2 \exp_2^*[W_b[f]](\xi) + v(\xi). \tag{117}$$

6. Existence of solutions of the band-limited equation.

In this section, we give conditions under which the sequence $\{\psi_n\}_{n=0}^{\infty}$ of fixed point iterates for (116) obtained by using the function v defined by (115) as an initial approximation converges. More explicitly, ψ_0 is defined by the formula

$$\psi_0(\xi) = v(\xi), \tag{118}$$

and for each integer $n \geqslant 0$, ψ_{n+1} is obtained from ψ_n via

$$\psi_{n+1}(\xi) = R[\psi_n](\xi). \tag{119}$$

Theorem 9. *Suppose that $\lambda > 0$ is a real number, and that $v \in L^1(\mathbb{R})$ such that*

$$\|v\|_1 \leqslant \frac{\pi}{2}\lambda^2. \tag{120}$$

Then the sequence $\{\psi_n\}_{n=0}^{\infty}$ defined by (118) and (119) converges in $L^1(\mathbb{R})$ norm to a function ψ which satisfies the equation (116) for almost all $\xi \in \mathbb{R}$. If v is a continuous function, then ψ is also continuous. If, in addition to being an element of $L^1(\mathbb{R})$ such that (120) holds, v is an element of $L^\infty(\mathbb{R})$ and

$$\|v\|_\infty \leqslant \frac{\pi}{2}\lambda^2, \tag{121}$$

then the sequence ψ_n also converges to ψ in $L^\infty(\mathbb{R})$ norm (i.e., uniformly).

Proof. We observe that the Fréchet derivative (see Section 2.5) of R at f is the linear operator $R'_f : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ given by the formula

$$R'_f[h](\xi) = \frac{\widetilde{W}_b[f] * \widetilde{W}_b[h](\xi)}{4\pi} - \frac{2\lambda^2}{\pi}\exp_1^*[W_b[f]] * W_b[h](\xi). \tag{122}$$

From formulas (107) and (108) and the definition of $b(\xi)$ we see that

$$\|W_b[f]\|_1 \leqslant \frac{\|f\|_1}{2\lambda^2}, \tag{123}$$

and

$$\left\|\widetilde{W}_b[f]\right\|_1 \leqslant \frac{\|f\|_1}{\sqrt{2}\lambda} \tag{124}$$

for all $f \in L^1(\mathbb{R})$. From (122), (123) and (124) we conclude that

$$\begin{aligned} \left\|R'_f[h]\right\|_1 &\leqslant \frac{1}{4\pi}\left\|\widetilde{W}_b[f]\right\|_1\left\|\widetilde{W}_b[h]\right\|_1 + \frac{2\lambda^2}{\pi}\|W_b[f]\|_1 \exp\left(\frac{\|W_b[f]\|_1}{2\pi}\right)\|W_b[h]\|_1 \\ &\leqslant \frac{\|f\|_1\|h\|_1}{8\pi\lambda^2} + \frac{2\lambda^2}{\pi}\frac{\|f\|_1}{2\lambda^2}\frac{\|h\|_1}{2\lambda^2}\exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right) \\ &\leqslant \left(\frac{\|f\|_1}{8\pi\lambda^2} + \frac{\|f\|_1}{2\pi\lambda^2}\exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right)\right)\|h\|_1 \end{aligned} \tag{125}$$

for all f and h in $L^1(\mathbb{R})$. Similarly, by combining (117), (123) and (124) we conclude that

$$\begin{aligned}\|R[f]\|_1 &\leqslant \frac{1}{8\pi}\left\|\widetilde{W}_b[f]\right\|_1^2 + \frac{\lambda^2}{\pi}\|W_b[f]\|_1^2 \exp\left(\frac{\|W_b[f]\|_1}{2\pi}\right) + \|v\|_1 \\ &\leqslant \frac{\|f\|_1^2}{16\pi\lambda^2} + \frac{\|f\|_1^2}{4\pi\lambda^2}\exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right) + \|v\|_1\end{aligned} \tag{126}$$

whenever $f \in L^1(\mathbb{R})$. We now let $r = \pi\lambda^2$ and denote by B the closed ball of radius r centered at 0 in $L^1(\mathbb{R})$. Suppose that $f \in L^1(\mathbb{R})$ such that

$$\|f\|_1 \leqslant r = \pi\lambda^2, \tag{127}$$

and that

$$\|v\|_1 \leqslant \frac{r}{2} = \frac{\pi\lambda^2}{2}. \tag{128}$$

We insert (127) and (128) into (126) in order to obtain

$$\begin{aligned}\|R[f]\|_1 &\leqslant \frac{r^2}{16\pi\lambda^2} + \frac{r^2}{4\pi\lambda^2}\exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right) + \frac{r}{2} \\ &= \left(\frac{r}{16\pi\lambda^2} + \frac{r}{4\pi\lambda^2}\exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right) + \frac{1}{2}\right) r \\ &= \left(\frac{1}{16} + \frac{1}{4}\exp\left(\frac{1}{4}\right) + \frac{1}{2}\right) r \\ &\leqslant \frac{9}{10} r,\end{aligned} \tag{129}$$

from which we conclude that R maps B into itself. Next, we insert (127) into (125) in order to obtain

$$\left\|R'_f[h]\right\|_1 \leqslant \left(\frac{1}{8} + \frac{1}{2}\exp\left(\frac{1}{4}\right)\right)\|h\|_1 \leqslant \frac{8}{10}\|h\|_1, \tag{130}$$

which shows that R is a contraction on B. We now invoke the contraction mapping theorem (Theorem 4 in Section 2.5) in order to conclude that any sequence of fixed point iterates for (116), which originates in B will converge in $L^1(\mathbb{R})$ to a solution of (116). Since $\{\psi_n\}$ is such a sequence, it converges in $L^1(\mathbb{R})$ to a function ψ such that

$$R[\psi](x) = \psi(x) \quad \text{for almost all } \ x \in \mathbb{R}. \tag{131}$$

If $f \in L^1(\mathbb{R})$, then

$$\begin{aligned}\left|\exp_2^*[f](\eta) - \exp_2^*[f](\xi)\right| &\leqslant 4\lambda^2\left(\frac{\|f\|_1}{2!(2\pi)} + \frac{\|f\|_1^2}{3!(2\pi)^2} + \cdots\right)\int_{-\infty}^{\infty}|f(\eta-y) - f(\xi-y)|\ dy \\ &\leqslant 4\lambda^2\exp\left(\|f\|_\infty\right)\int_{-\infty}^{\infty}|f(\eta-y) - f(\xi-y)|\ dy.\end{aligned} \tag{132}$$

The dominated convergence theorem implies that

$$\int_{-\infty}^{\infty}|f(\eta-y) - f(\xi-y)|\ dy \to 0 \quad \text{as} \quad |\eta-\xi| \to 0. \tag{133}$$

From (132) and (133), we see that $\exp_2^*[f]$ is continuous whenever $f \in L^1(\mathbb{R})$. A nearly identical argument shows that $f * f$ is continuous whenever $f \in L^1(\mathbb{R})$. We conclude from these observations and the definition of the operator R that $R[f]$ is continuous when v is continuous and $f \in L^1(\mathbb{R})$. In particular, $\psi = R[\psi]$ is continuous.

Now suppose that in addition to being an element of $L^1(\mathbb{R})$ such that $\|v\|_1 \leqslant r/2$, v is an element of

$L^\infty(\mathbb{R})$ such that (121) holds; that is,

$$\|v\|_\infty \leqslant \frac{\pi\lambda^2}{2} = \frac{r}{2} \tag{134}$$

We observe that formulas (107) and (108) and the definition of $b(\xi)$ imply that

$$\|W_b[f]\|_\infty \leqslant \frac{\|f\|_\infty}{2\lambda^2} \tag{135}$$

and

$$\left\|\widetilde{W}_b[f]\right\|_\infty \leqslant \frac{\|f\|_\infty}{\sqrt{2}\lambda} \tag{136}$$

for all $f \in L^\infty(\mathbb{R})$. If $\|f\|_\infty \leqslant r$ and $\|f\|_1 \leqslant r$, then using the definitions of R, (123), (124) (135), and (136) we obtain

$$\begin{aligned}\|R[f]\|_\infty &\leqslant \frac{1}{8\pi}\left\|\widetilde{W}_b[f]\right\|_\infty\left\|\widetilde{W}_b[f]\right\|_1 + \frac{\lambda^2}{\pi}\|W_b[f]\|_\infty\|W_b[f]\|_1\exp\left(\frac{\|W_b[f]\|_1}{2\pi}\right) + \|v\|_\infty \\ &\leqslant \frac{r^2}{16\pi\lambda^2} + \frac{r^2}{4\pi\lambda^2}\exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right) + \frac{r}{2} \\ &\leqslant \frac{9}{10}r.\end{aligned} \tag{137}$$

Similarly, if $\|f\|_\infty \leqslant r$ and $\|f\|_1 \leqslant r$, then from the definitions of R', (123), (124) (135), and (136) we conclude that

$$\begin{aligned}\left\|R'_f[h]\right\|_\infty &\leqslant \frac{1}{4\pi}\left\|\widetilde{W}_b[f]\right\|_1\left\|\widetilde{W}_b[h]\right\|_\infty + \frac{2\lambda^2}{\pi}\|W_b[f]\|_1\exp\left(\frac{\|W_b[f]\|_1}{2\pi}\right)\|W_b[h]\|_\infty \\ &\leqslant \left(\frac{\|f\|_1}{8\pi\lambda^2} + \frac{\|f\|_1}{2\pi\lambda^2}\exp\left(\frac{\|f\|_1}{4\pi\lambda^2}\right)\right)\|h\|_\infty \\ &\leqslant \left(\frac{1}{8} + \frac{1}{2}\exp\left(\frac{1}{4}\right)\right)\|h\|_1 \leqslant \frac{8}{10}\|h\|_\infty\end{aligned} \tag{138}$$

We conclude from (137) that

$$\|\psi_n\|_\infty \leqslant r \tag{139}$$

for all positive integers n. From this observation and (138), we conclude that the sequence $\{\psi_n\}$ is Cauchy in $L^\infty(\mathbb{R})$ norm. It follows that ψ_n converges to ψ in $L^\infty(\mathbb{R})$ as well as in $L^1(\mathbb{R})$. □

If ψ is a solution of (116) then the function σ_b defined by the formula

$$\sigma_b(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ix\xi)\psi(\xi)\,d\xi \tag{140}$$

is clearly a solution of the band-limited integral equation (106). We note that because $\psi \in L^1(\mathbb{R})$, the integral in (140) is well-defined and σ_b is an element of the space $C_0(\mathbb{R})$ of continuous functions which vanish at infinity. We record this observation as follows:

Theorem 10. *Suppose that $\lambda > 0$ is a real number. Suppose also that $p \in L^1(\mathbb{R})$ such that $\widehat{p} \in L^1(\mathbb{R})$ and*

$$\|\widehat{p}\|_1 < \frac{\pi}{2}\lambda^2. \tag{141}$$

Then there exists a function $\sigma_b \in C_0(\mathbb{R})$ which is a solution of the integral equation (106).

Remark 1. *Since σ_b is not necessarily in $L^1(\mathbb{R})$, the integral*

$$\int_{-\infty}^{\infty}\exp(-ix\xi)\sigma_b(x)\,dx \tag{142}$$

need not exist. Nor is the existence of the improper integral

$$\lim_{R\to\infty}\int_{-R}^{R}\exp(-ix\xi)\sigma_b(x)\,dx \tag{143}$$

guaranteed. However, when viewed as a tempered distribution, the Fourier transform of σ_b exists and is ψ; that is to say,

$$\int_{-\infty}^{\infty}\psi(x)f(x)\,dx = \int_{-\infty}^{\infty}\sigma_b(x)\widehat{f}(x)\,dx \tag{144}$$

for all functions $f \in S(\mathbb{R})$. In the next section we will prove that under additional assumptions on v, ψ lies in $L^2(\mathbb{R})$. This implies that $\sigma_b \in L^2(\mathbb{R})$ and by so doing ensures the convergence of the improper Riemann integral (143).

7. Fourier estimate

In this section, we derive a pointwise estimate on the solution ψ of Equation (116) under additional assumptions on the function v.

Lemma 1. *Suppose that a and C are real numbers such that*

$$0 \leqslant C < a. \tag{145}$$

Suppose also that $f \in L^1(\mathbb{R})$, and that

$$|f(\xi)| \leqslant C\exp(-a|\xi|) \quad \textit{for all} \quad \xi \in \mathbb{R}. \tag{146}$$

Then

$$\left|\exp_2^*[f](\xi)\right| \leqslant \frac{C^2}{2\pi}\exp(-a|\xi|)\frac{1+a|\xi|}{a}\exp\left(\frac{C}{2\pi a}\right)\exp\left(\frac{C}{2\pi}|\xi|\right) \quad \textit{for all} \quad \xi \in \mathbb{R}, \tag{147}$$

where $\exp_2^*$ is the operator defined in (113).

Proof. Let

$$g(\xi) = C\exp(-a|\xi|) \tag{148}$$

and for each integer $m > 0$, denote by g_m the m-fold convolution product of the function g. That is to say that g_1 is defined via the formula

$$g_1(\xi) = g(\xi) \tag{149}$$

and for each integer $m > 0$, g_{m+1} is defined in terms of g_m by the formula

$$g_{m+1}(\xi) = g_m * g(\xi). \tag{150}$$

We observe that for each integer $m > 0$ and all $\xi \in \mathbb{R}$,

$$g_m(\xi) = 2\sqrt{aC}\left(\frac{C|\xi|}{2\pi}\right)^{m-1/2}\frac{K_{m-1/2}(a|\xi|)}{\Gamma(m)}, \tag{151}$$

where K_ν denotes the modified Bessel function of the second kind of order ν (see Section 2.3). By repeatedly applying Theorem 1 of Section 2.3, we conclude that for all integers $m > 0$ and all real t,

$$\begin{aligned} K_{m-1/2}(t) &\leqslant K_{1/2}(t)\prod_{j=1}^{m-1}\left(\frac{2\left(j-\frac{1}{2}\right)}{t}+1\right) \\ &= K_{1/2}(t)\left(\frac{2}{t}\right)^{m-1}\frac{\Gamma\left(\frac{1+t}{2}+m-1\right)}{\Gamma\left(\frac{1+t}{2}\right)}. \end{aligned} \tag{152}$$

We insert the identity

$$K_{1/2}(t) = \sqrt{\frac{\pi}{2t}} \exp(-t) \tag{153}$$

into (151) in order to conclude that for all integers $m > 0$ and all real numbers $t > 0$,

$$K_{m-1/2}(t) \leqslant \frac{\sqrt{\pi}}{2} \left(\frac{t}{2}\right)^{1/2-m} \exp(-t) \frac{\Gamma\left(\frac{1+t}{2} + m - 1\right)}{\Gamma\left(\frac{1+t}{2}\right)}. \tag{154}$$

By combining (154) and (151) we conclude that

$$g_m(\xi) \leqslant C \exp(-a|\xi|) \left(\frac{C}{\pi a}\right)^{m-1} \frac{\Gamma\left(\frac{1+a|\xi|}{2} + m - 1\right)}{\Gamma(m)\Gamma\left(\frac{1+a|\xi|}{2}\right)} \tag{155}$$

for all integers $m > 0$ and all $\xi \neq 0$. Moreover, the limit as $\xi \to 0$ of each side of (155) is finite and the two limits are equal, so (155) in fact holds for all $\xi \in \mathbb{R}$. We sum (155) over $m = 2, 3, \ldots$ in order to conclude that

$$\begin{aligned}\exp_2^*[g](\xi) &\leqslant C \exp(-a|\xi|) \sum_{m=2}^{\infty} \left(\frac{C}{\pi a}\right)^{m-1} \frac{\Gamma\left(\frac{1+a|\xi|}{2} + m - 1\right)}{\Gamma(m+1)\Gamma(m)\Gamma\left(\frac{1+a|\xi|}{2}\right)} \\ &= C \exp(-a|\xi|) \sum_{m=1}^{\infty} \left(\frac{C}{\pi a}\right)^{m} \frac{\Gamma\left(\frac{1+a|\xi|}{2} + m\right)}{\Gamma(m+2)\Gamma(m+1)\Gamma\left(\frac{1+a|\xi|}{2}\right)}\end{aligned} \tag{156}$$

for all $\xi \in \mathbb{R}$. Now we observe that

$$\frac{1}{\Gamma(m+2)} \leqslant \left(\frac{1}{2}\right)^m \quad \text{for all } \ m = 0, 1, 2, \ldots. \tag{157}$$

Inserting (157) into (156) yields

$$\exp_2^*[g](\xi) \leqslant C \exp(-a|\xi|) \sum_{m=1}^{\infty} \left(\frac{C}{2\pi a}\right)^{m} \frac{\Gamma\left(\frac{1+a|\xi|}{2} + m\right)}{\Gamma(m+1)\Gamma\left(\frac{1+a|\xi|}{2}\right)} \tag{158}$$

for all $\xi \in \mathbb{R}$. Now we apply the binomial theorem (Theorem 2 of Section 2.4), which is justified since $C < a < 2\pi a$, to conclude that

$$\begin{aligned}\exp_2^*[g](\xi) &\leqslant C \exp(-a|\xi|) \left(\left(1 - \frac{C}{2\pi a}\right)^{-\frac{1+a|\xi|}{2}} - 1\right) \\ &= C \exp(-a|\xi|) \left(\exp\left(\frac{1+a|\xi|}{2} \log\left(\frac{1}{1 - \frac{C}{2\pi a}}\right)\right) - 1\right)\end{aligned} \tag{159}$$

for all $\xi \in \mathbb{R}$. We observe that

$$\exp(x) - 1 \leqslant x \exp(x) \quad \text{for all } \ x \geqslant 0, \tag{160}$$

and

$$1 \leqslant \log\left(\frac{1}{1-x}\right) \leqslant 2x \quad \text{for all } \ 0 \leqslant x \leqslant \frac{1}{2\pi}. \tag{161}$$

By combining (160) and (161) with (159) we conclude that

$$\begin{aligned}\exp_2^*[g](\xi) &\leqslant C\exp(-a|\xi|)\frac{1+a|\xi|}{2}\log\left(\frac{1}{1-\frac{C}{2\pi a}}\right)\exp\left(\frac{1+a|\xi|}{2}\log\left(\frac{1}{1-\frac{C}{2\pi a}}\right)\right)\\ &\leqslant \frac{C^2}{2\pi}\exp(-a|\xi|)\frac{1+a|\xi|}{a}\exp\left(\frac{C}{2\pi a}\right)\exp\left(\frac{C}{2\pi}|\xi|\right)\end{aligned} \tag{162}$$

for all $\xi \in \mathbb{R}$. Note that in (162), we used the assumption that $C < a$ in order to apply the inequality (161). Owing to (146),

$$|\exp_2^*[f](\xi)| \leqslant \exp_2^*[g](\xi) \quad \text{for all } \xi \in \mathbb{R}. \tag{163}$$

By combining this observation with (162), we obtain (147), which completes the proof. □

Remark 2. *Kummer's confluent hypergeometric function $M(a,b,z)$ is defined by the series*

$$M(a,b,z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \frac{(a)_3 z^3}{(b)_3 3!} + \cdots, \tag{164}$$

where $(a)_n$ is the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\ldots(a+n-1). \tag{165}$$

By comparing the definition of $M(a,b,z)$ with (156), we conclude that

$$|\exp_2^*[f](\xi)| \leqslant C\exp(-a|\xi|)\left(M\left(\frac{1+a|\xi|}{2}, 2, \frac{C}{\pi a}\right) - 1\right) \quad \textit{for all } \xi \in \mathbb{R} \tag{166}$$

provided

$$|f(\xi)| \leqslant C\exp(-a|\xi|) \quad \textit{for all } \xi \in \mathbb{R}. \tag{167}$$

The weaker bound (147) is sufficient for our immediate purposes, but formula (166) might serve as a basis for improved estimates on solutions of Kummer's equation.

The following lemma is a special case of Formula (151).

Lemma 2. *Suppose that $C \geqslant 0$ and $a > 0$ are real numbers, and that $f \in L^1(\mathbb{R})$ such that*

$$|f(\xi)| \leqslant C\exp(-a|\xi|) \quad \textit{for all } \xi \in \mathbb{R}. \tag{168}$$

Then

$$|f * f(\xi)| \leqslant C^2\exp(-a|\xi|)\left(\frac{1+a|\xi|}{a}\right) \quad \textit{for all } \xi \in \mathbb{R}. \tag{169}$$

We will also make use of the following elementary observation.

Lemma 3. *Suppose that $a > 0$ is a real number. Then*

$$\exp(-a|\xi|)|\xi| \leqslant \frac{1}{a\exp(1)} \quad \textit{for all } \xi \in \mathbb{R}. \tag{170}$$

We combine Lemmas 1 and 2 with (123) and (124) in order to obtain the following key estimate.

Theorem 11. *Suppose that $\Gamma > 0$, $\lambda > 0$, $a > 0$ and $C \geqslant 0$ are real numbers such that*

$$0 \leqslant C < 2a\lambda^2. \tag{171}$$

Suppose also that $f \in L^1(\mathbb{R})$ such that

$$|f(\xi)| \leqslant C\exp(-a|\xi|) \quad \textit{for all } |\xi| \leqslant \sqrt{2}\lambda, \tag{172}$$

and that $v \in L^1(\mathbb{R})$ such that

$$|v(\xi)| \leqslant \Gamma \exp(-a|\xi|) \quad \textit{for all} \;\; \xi \in \mathbb{R}. \tag{173}$$

Suppose further that R is the operator defined via (117). Then

$$|R[f](\xi)| \leqslant \exp(-a|\xi|)\left(\frac{C^2}{\lambda^2}\left(\frac{1+a|\xi|}{a}\right)\left(\frac{1}{16\pi}+\frac{1}{2\pi}\exp\left(\frac{C}{4\pi\lambda^2 a}\right)\exp\left(\frac{C}{4\pi\lambda^2}|\xi|\right)\right)+\Gamma\right) \tag{174}$$

for all $\xi \in \mathbb{R}$.

Proof. We define the operator R_1 via the formula

$$R_1[f](\xi) = \frac{1}{8\pi}\widetilde{W}_b[f] * \widetilde{W}_b[f](\xi) \tag{175}$$

and R_2 by the formula

$$R_2[f](\xi) = -4\lambda^2 \exp_2^*[W_b[f]](\xi), \tag{176}$$

where W_b and $\widetilde{W}_b$ are defined as in Section 5. Then

$$R[f](\xi) = R_1[f](\xi) + R_2[f](\xi) + v(\xi) \tag{177}$$

for all $\xi \in \mathbb{R}$. We observe that

$$\left|\widetilde{W}_b[f](\xi)\right| \leqslant \frac{C}{\sqrt{2}\lambda}\exp(-a|\xi|) \quad \text{for all} \;\; \xi \in \mathbb{R}. \tag{178}$$

By combining Lemma 2 with (178) we obtain

$$|R_1[f](\xi)| \leqslant \frac{C^2}{16\pi\lambda^2}\exp(-a|\xi|)\left(\frac{1+a|\xi|}{a}\right) \quad \text{for all} \;\; \xi \in \mathbb{R}. \tag{179}$$

Now we observe that

$$|W_b[f](\xi)| \leqslant \frac{C}{2\lambda^2}\exp(-a|\xi|) \quad \text{for all} \;\; \xi \in \mathbb{R}. \tag{180}$$

Combining Lemma 1 with (180) yields

$$|R_2[f](\xi)| \leqslant \frac{C^2}{2\pi\lambda^2}\exp(-a|\xi|)\left(\frac{1+a|\xi|}{a}\right)\exp\left(\frac{C}{4\pi\lambda^2 a}\right)\exp\left(\frac{C}{4\pi\lambda^2}|\xi|\right) \tag{181}$$

for all $\xi \in \mathbb{R}$. Note that (171) ensures that the hypothesis (145) in Lemma 1 is satisfied. We combine (179) with (181) and (173) in order to obtain (174), and by so doing we complete the proof. ☐

Remark 3. *Note that Theorem 11 only requires that $f(\xi)$ satisfy a bound on the interval $[-\sqrt{2}\lambda, \sqrt{2}\lambda]$ and not on the entire real line.*

In the next theorem, we use Theorem 11 to bound the solution of (116) under an assumption on the decay of v.

Theorem 12. *Suppose that $\lambda > 0$, $a > 0$ and $\Gamma \geqslant 0$ are real numbers such that*

$$\lambda \geqslant 2\max\left\{\Gamma, \frac{1}{a}\right\}. \tag{182}$$

Suppose also that $v \in L^1(\mathbb{R})$ such that

$$|v(\xi)| \leqslant \Gamma \exp(-a|\xi|) \quad \textit{for all} \;\; \xi \in \mathbb{R}. \tag{183}$$

Then there exists a solution ψ of (116) such that

$$|\psi(\xi)| < 2\Gamma \exp(-a|\xi|) \quad \textit{for almost all} \quad |\xi| \leqslant \sqrt{2}\lambda \tag{184}$$

and

$$|\psi(\xi)| < 2\Gamma \exp\left(-\left(a - \frac{1}{\lambda}\right)|\xi|\right) \quad \textit{for all} \;\; \xi \in \mathbb{R}. \tag{185}$$

Proof. From (182) and (183) we obtain

$$\|v\|_1 \leqslant \Gamma \int \exp(-a\,|\xi|)\, d\xi = \frac{\Gamma}{a} < \frac{\lambda^2}{4}. \tag{186}$$

It follows from Theorem 9 and (186) that a solution $\psi(\xi)$ of (116) is obtained as the limit of the sequence of fixed point iterates $\{\psi_n(\xi)\}$ defined by the formula

$$\psi_0(\xi) = v(\xi) \tag{187}$$

and the recurrence

$$\psi_{n+1}(\xi) = R\,[\psi_n\,]\,(\xi). \tag{188}$$

We now derive pointwise estimates on the iterates $\psi_n(\xi)$ in order to establish (185).

We let $\{\beta_k\}_{k=0}^{\infty}$ be the sequence of real numbers be generated by the recurrence relation

$$\beta_{k+1} = \frac{\beta_k^2}{2\lambda} + \Gamma \tag{189}$$

with the initial value

$$\beta_0 = \Gamma. \tag{190}$$

From (182) we see that if $\beta_k < 2\Gamma$ then

$$\beta_{k+1} = \frac{\beta_k^2}{2\lambda} + \Gamma \leqslant \frac{4\Gamma^2}{2\lambda} + \Gamma \leqslant 2\Gamma. \tag{191}$$

It follows by induction that $\beta_k \leqslant 2\Gamma$ for all $k \geqslant 0$. The sequence β_k is monotonically increasing, so it in fact converges to a limit $\beta < 2\Gamma$.

Now suppose that $n \geqslant 0$ is an integer, and that

$$|\psi_n(\xi)| \leqslant \beta_n \exp(-a|\xi|) \quad \text{for all} \quad |\xi| \leqslant \sqrt{2}\lambda. \tag{192}$$

When $n = 0$, this is simply the assumption (183). The function $\psi_{n+1}(\xi)$ is obtained from $\psi_n(\xi)$ via the formula

$$\psi_{n+1}(\xi) = R\,[\psi\,]\,(\xi). \tag{193}$$

We combine Theorem 11 with (193) and (192) to conclude that

$$|\psi_{n+1}(\xi)| \leqslant \exp(-a|\xi|)\left(\frac{\beta_n^2}{\lambda^2}\left(\frac{1+a|\xi|}{a}\right)\left(\frac{1}{16\pi} + \frac{1}{2\pi}\exp\left(\frac{\beta_n}{4\pi\lambda^2 a}\right)\exp\left(\frac{\beta_n}{4\pi\lambda^2}|\xi|\right)\right) + \Gamma\right) \tag{194}$$

for all $\xi \in \mathbb{R}$. The hypothesis (171) of Theorem 11 is satisfied since

$$\beta_n \leqslant 2\Gamma \leqslant 2\lambda^2 a \tag{195}$$

for all integers $n \geqslant 0$. We restrict ξ to the interval $[-\sqrt{2}\lambda, \sqrt{2}\lambda]$ in (194) and use the fact that

$$\frac{1}{a\lambda} < \frac{1}{2} \tag{196}$$

which is a consequence of (182), in order to conclude that

$$\begin{aligned} |\psi_{n+1}(\xi)| &\leqslant \exp(-a|\xi|)\left(\frac{\beta_n^2}{\lambda^2}\left(\frac{1+a\sqrt{2}\lambda}{a}\right)\left(\frac{1}{16\pi} + \frac{1}{2\pi}\exp\left(\frac{\beta_n}{4\pi\lambda^2 a}\right)\exp\left(\frac{\beta_n}{4\pi\lambda^2}\sqrt{2}\lambda\right)\right) + \Gamma\right) \\ &\leqslant \exp(-a|\xi|)\left(\frac{\beta_n^2}{\lambda}\left(\frac{1}{2} + \sqrt{2}\right)\left(\frac{1}{16\pi} + \frac{1}{2\pi}\exp\left(\frac{\beta_n}{8\pi\lambda}\right)\exp\left(\frac{\beta_n}{2\sqrt{2}\pi\lambda}\right)\right) + \Gamma\right) \end{aligned} \tag{197}$$

for all $|\xi| \leqslant \sqrt{2}\lambda$. Now we combine (197) with the inequality

$$\frac{\beta_n}{\lambda} \leqslant \frac{2\Gamma}{\lambda} \leqslant 1 \tag{198}$$

and the observation that

$$\left(\frac{1}{2}+\sqrt{2}\right)\left(\frac{1}{16\pi}+\frac{1}{2\pi}\exp\left(\frac{1}{8\pi}\right)\exp\left(\frac{1}{2\sqrt{2}\pi}\right)\right) < \frac{1}{2} \tag{199}$$

in order to conclude that

$$\begin{aligned} |\psi_{n+1}(\xi)| &< \left(\frac{\beta_n^2}{2\lambda}+\Gamma\right)\exp(-a|\xi|) \\ &= \beta_{n+1}\exp(-a|\xi|) \end{aligned} \tag{200}$$

for all $|\xi| \leqslant \sqrt{2}\lambda$. We conclude by induction that (192) holds for all integers $n \geqslant 0$.

The sequence $\{\psi_n(\xi)\}$ converges to $\psi(\xi)$ in $L^1(\mathbb{R})$ norm and so a subsequence of $\psi_n(\xi)$ converges to $\psi(\xi)$ pointwise almost everywhere. Since (192) holds for all integers $n \geqslant 0$ and $\beta_{n+1} < 2\Gamma$ for all nonnegative integers n, we conclude that

$$|\psi(\xi)| < 2\Gamma\exp(-a|\xi|) \tag{201}$$

for almost all $|\xi| \leqslant \sqrt{2}\lambda$.

We now apply Theorem 11 to the function $\psi(\xi)$ (which is justified since $2\Gamma < 2\lambda^2 a$) to conclude that

$$|\psi(\xi)| \leqslant \exp(-a|\xi|)\left(\frac{4\Gamma^2}{\lambda^2}\left(\frac{1+a|\xi|}{a}\right)\left(\frac{1}{16\pi}+\frac{1}{2\pi}\exp\left(\frac{2\Gamma}{4\pi\lambda^2 a}\right)\exp\left(\frac{2\Gamma}{4\pi\lambda^2}|\xi|\right)\right)+\Gamma\right) \tag{202}$$

for all $\xi \in \mathbb{R}$. Note the distinction between (192) and (202) is that the former only holds for almost all ξ in the interval $[-\sqrt{2}\lambda, \sqrt{2}\lambda]$, while the later holds for all ξ on the real line. It follows from (182) that

$$\frac{1}{\lambda a} < \frac{1}{2} \quad \text{and} \quad \frac{\Gamma}{\lambda} < \frac{1}{2} \tag{203}$$

We insert these bounds into (202) in order to conclude that

$$|\psi(\xi)| \leqslant \Gamma\exp(-a|\xi|)\left(\left(1+\frac{2|\xi|}{\lambda}\right)\left(\frac{1}{16\pi}+\frac{1}{2\pi}\exp\left(\frac{1}{8\pi}\right)\exp\left(\frac{1}{4\pi\lambda}|\xi|\right)\right)+1\right) \tag{204}$$

for all $\xi \in \mathbb{R}$. We conclude from Lemma 3 that

$$\exp\left(-\frac{1}{2\lambda}|\xi|\right)\left(1+\frac{2|\xi|}{\lambda}\right) \leqslant 1+4\exp(-1) \quad \text{for all } \ \xi \in \mathbb{R}. \tag{205}$$

Moreover, we observe that

$$\exp\left(-\frac{1}{2\lambda}|\xi|\right)\left(\frac{1}{16\pi}+\frac{1}{2\pi}\exp\left(\frac{1}{8\pi}\right)\exp\left(\frac{1}{4\pi\lambda}|\xi|\right)\right) \leqslant \left(\frac{1}{16\pi}+\frac{1}{2\pi}\exp\left(\frac{1}{8\pi}\right)\right) \tag{206}$$

for all $\xi \in \mathbb{R}$. We conclude from (205), (206) and the fact that

$$\exp\left(-\eta\,|\xi|\right) \leqslant 1 \quad \text{for all } \ \eta, \xi \in \mathbb{R} \tag{207}$$

that

$$\begin{aligned} &\exp\left(-\frac{1}{2\lambda}\,|\xi|\right)\left(\left(1+\frac{4|\xi|}{\lambda}\right)\left(\frac{1}{16\pi}+\frac{1}{2\pi}\exp\left(\frac{1}{8\pi}\right)\exp\left(\frac{1}{4\pi\lambda}|\xi|\right)\right)+1\right) \\ &\leqslant (1+4\exp(-1))\left(\frac{1}{16\pi}+\frac{1}{2\pi}\exp\left(\frac{1}{8\pi}\right)\right)+1 \\ &< 2 \end{aligned} \tag{208}$$

for all $\xi \in \mathbb{R}$. By inserting (208) into (202), we conclude that

$$|\psi(\xi)| < 2\Gamma \exp\left(-\left(a - \frac{1}{\lambda}\right)|\xi|\right) \quad \text{for all } \xi \in \mathbb{R}, \tag{209}$$

which is (185). □

Suppose that $\psi \in L^1(\mathbb{R})$ is a solution of (116). Then the function σ_b defined by the formula

$$\sigma_b(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ix\xi)\psi(\xi)\, d\xi \tag{210}$$

is a solution of the integral equation (106). However, the Fourier transform of (210) might only be defined in the sense of tempered distributions and not as a Lebesgue or improper Riemann integral. If, however, we assume the function p appearing in (116) is an element of $L^1(\mathbb{R})$ and impose the hypotheses of Theorem 12 on the Fourier transform of p, then the Fourier transform of $\widehat{\sigma_b}$ decays faster than any polynomial. From this, we conclude that $\sigma_b \in L^2(\mathbb{R})$ and that σ_b is infinitely differentiable. In this event, there is no difficulty in defining the Fourier transform of σ_b. Moreover, since $\widehat{p}$ is continuous in this case, Theorem 9 ensures that $\widehat{\sigma_b}$ is also continuous.

We record these observations in the following theorem.

Theorem 13. *Suppose that there exist real numbers $\lambda > 0$, $\Gamma > 0$ and $a > 0$ such that*

$$\lambda > 2\max\left\{\Gamma, \frac{1}{a}\right\}. \tag{211}$$

Suppose also that p is an element of $L^1(\mathbb{R})$ such that

$$|\widehat{p}(\xi)| \leqslant \Gamma \exp(-a|\xi|) \quad \textit{for all } \xi \in \mathbb{R}. \tag{212}$$

Then there exists a solution $\sigma_b \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ of the integral equation (106) such that $\widehat{\sigma_b}$ is a continuous function,

$$|\widehat{\sigma_b}(\xi)| < 2\Gamma \exp(-a|\xi|) \quad \textit{for all } |\xi| \leqslant \sqrt{2}\lambda, \tag{213}$$

and

$$|\widehat{\sigma_b}(\xi)| < 2\Gamma \exp\left(-\left(a - \frac{1}{\lambda}\right)|\xi|\right) \quad \textit{for all } \xi \in \mathbb{R}. \tag{214}$$

8. Solution of the perturbed integral equation

We would like to insert the solution σ_b of (106) into the original equation (73) in order to construct a function ν of small magnitude such that

$$\sigma_b(x) = S[T[\sigma_b]](x) + p(x) + \nu(x). \tag{215}$$

However, there is no guarantee that the integral

$$\frac{1}{2\lambda} \int_{-\infty}^{\infty} \sin(2\lambda|x-y|)\,\sigma(y)\, dy \tag{216}$$

defining the function $T[\sigma_b]$ exists, nor are we assured that the expression

$$\frac{\widehat{\sigma}(\xi)}{4\lambda^2 - \xi^2}, \tag{217}$$

which is formally the Fourier transform of $T[\sigma]$, defines a tempered distribution.

To remedy this problem, we define a “band-limited” version σ of σ_b by the formula

$$\widehat{\sigma}(\xi) = \widehat{\sigma_b}(\xi)b(\xi), \tag{218}$$

where $b(\xi)$ is the function used to define the operator T_b. We observe that there is no difficulty in defining $T[\sigma]$ — it is the inverse Fourier transform of the function

$$\frac{\widehat{\sigma_b}(\xi)b(\xi)}{4\lambda^2 - \xi^2}, \tag{219}$$

which is in $L^1(\mathbb{R})$ since $\widehat{\sigma_b}$ and

$$\frac{b(\xi)}{4\lambda^2 - \xi^2} \tag{220}$$

are both square integrable. Moreover, $T_b[\sigma_b] = T[\sigma]$, so that

$$\sigma_b(x) = S[T[\sigma]](x) + p(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{221}$$

Rearranging (221), we obtain

$$\sigma(x) = S[T[\sigma]](x) + p(x) + \nu(x) \quad \text{for all} \quad x \in \mathbb{R}, \tag{222}$$

where ν is defined the formula

$$\nu(x) = \sigma(x) - \sigma_b(x). \tag{223}$$

Using (214) and (223), we conclude that under the hypotheses of Theorem 13,

$$\begin{aligned} \|\nu\|_\infty &\leqslant \|\widehat{\sigma} - \widehat{\sigma_b}\|_1 \\ &< 2\Gamma \int_{|\xi| \geqslant \lambda} \exp\left(-\left(a - \frac{1}{\lambda}\right)|\xi|\right) d\xi \\ &\leqslant \frac{4\Gamma}{a - \frac{1}{\lambda}} \exp\left(-\left(a - \frac{1}{\lambda}\right)\lambda\right) \\ &\leqslant \frac{4\Gamma\exp(1)}{a} \exp(-a\lambda) \\ &\leqslant \frac{12\Gamma}{a} \exp(-a\lambda) \end{aligned} \tag{224}$$

By combining Theorem 13 with (224) we obtain the following.

Theorem 14. *Suppose that $q \in C^\infty(\mathbb{R})$ is a strictly positive, and that $x(t)$ is defined by the formula*

$$x(t) = \int_0^t \sqrt{q(u)}\, du. \tag{225}$$

Suppose also that $p(x)$ is defined via the formula

$$p(x) = 2\{t, x\}; \tag{226}$$

that is, $p(x)$ is twice the Schwarzian derivative of the variable t with respect to the variable x defined via (225). Suppose furthermore that there exist positive real numbers λ, Γ and a such that

$$\lambda \geqslant \max\left\{\frac{2}{a}, 2\Gamma, \sqrt{\frac{2\Gamma}{\pi}}\right\} \tag{227}$$

and

$$|\widehat{p}(\xi)| \leqslant \Gamma \exp(-a|\xi|) \quad \textit{for all} \;\; \xi \in \mathbb{R}. \tag{228}$$

Then there exist functions ν and σ in $L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that $\widehat{\sigma}$ and $\widehat{\nu}$ are elements of $L^2(\mathbb{R}) \cap C(\mathbb{R})$, σ is a solution of the nonlinear integral equation

$$\sigma(x) = S[T[\sigma]](x) + p(x) + \nu(x), \tag{229}$$

$$|\widehat{\sigma}(\xi)| < 2\Gamma \exp(-a|\xi|) \quad \textit{for all} \quad |\xi| \leqslant \sqrt{2}\lambda, \tag{230}$$

$$\widehat{\sigma}(\xi) = 0 \quad \textit{for all} \quad |\xi| > \sqrt{2}\lambda, \tag{231}$$

and

$$\|\nu\|_\infty \leqslant \frac{12\Gamma}{a} \exp(-a\lambda). \tag{232}$$

We now establish the principal result of this paper, Theorem 7, by approximating $\widehat{\sigma}$ with a sequence of $C_c(\mathbb{R})$ functions.

Proof of Theorem 7. First, we note that the hypotheses of Theorem 7 and those of Theorem 14 are identical and we denote by $\tilde{\sigma}$ and $\tilde{\nu}$ the functions obtained by invoking Theorem 14. As a compactly supported function in $L^2(\mathbb{R})$, $\widehat{\tilde{\sigma}}$ is in fact in $L^1(\mathbb{R})$. Consequently, there exists a sequence

$$\widehat{\psi}_1(\xi), \widehat{\psi}_2(\xi), \widehat{\psi}_3(\xi), \ldots \tag{233}$$

of $C_c^\infty(\mathbb{R})$ functions such that

$$\left\| \widehat{\psi_n} - \widehat{\tilde{\sigma}} \right\|_1 \to \ 0 \ \text{ as } \ n \to \infty. \tag{234}$$

Since $\widehat{\tilde{\sigma}}$ is also a compactly supported, continuous function, we can also ensure that

$$\left\| \widehat{\psi_n} - \widehat{\tilde{\sigma}} \right\|_\infty \to \ 0 \ \text{ as } \ n \to \infty. \tag{235}$$

Moreover, since the support of $\widehat{\tilde{\sigma}}$ is properly contained in $\left(-\sqrt{2}\lambda, \sqrt{2}\lambda\right)$ (see the definition $b(\xi)$) we can assume that the support of each of the functions $\widehat{\psi}_n$ is contained in the interval $(-\sqrt{2}\lambda, \sqrt{2}\lambda)$. The functions $\widehat{\psi}_n$ can be obtained, for instance, in the following fashion. We let

$$g(\xi) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{for all } |\xi| < 1 \\ 0 & \text{otherwise} \end{cases}, \tag{236}$$

and we define g_t for each $t > 0$ by the formula

$$g_t(\xi) = \frac{1}{t} g\left(\frac{\xi}{t}\right). \tag{237}$$

Then

$$g_t * \widehat{\tilde{\sigma}} \to \widehat{\tilde{\sigma}} \ \text{ as } \ t \to 0 \tag{238}$$

in $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$. By choosing a sufficiently large positive integer k and letting

$$\psi_n = g_{\frac{1}{k+n}} * \widehat{\tilde{\sigma}}, \tag{239}$$

we obtain a sequence of functions with support contained in $(-\sqrt{2}\lambda, \sqrt{2}\lambda)$ which converges to $\widehat{\sigma}$ in $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$.

For each positive integer n, we denote by ψ_n the inverse Fourier transform of $\widehat{\psi}_n$. Since the $\widehat{\psi_n}$ are compactly supported, infinitely differentiable functions (and therefore elements of the Schwartz space $S(\mathbb{R})$), the functions ψ_n are elements of $S(\mathbb{R})$. We observe that (234) implies that

$$\|\psi_n - \tilde{\sigma}\|_\infty \to 0 \ \text{ as } \ n \to \infty. \tag{240}$$

Moreover, since

$$\frac{1}{4\lambda^2 - \xi^2} \tag{241}$$

is bounded for all $\xi \in (-\sqrt{2}\lambda, \sqrt{2}\lambda)$, it follows from and (234) that

$$\left\| \frac{\widehat{\psi_n}(\xi) - \widehat{\tilde{\sigma}}(\xi)}{4\lambda^2 - \xi^2} \right\|_1 \to 0 \ \text{ as } \ n \to \infty \tag{242}$$

and

$$\left\| \frac{i\xi\widehat{\psi_n}(\xi) - i\xi\widehat{\tilde{\sigma}}(\xi)}{4\lambda^2 - \xi^2} \right\|_1 \to 0 \ \text{ as } \ n \to \infty. \tag{243}$$

From (242) and (243) and (61) we conclude that

$$\|T[\psi_n] - T[\tilde{\sigma}]\|_\infty \to 0 \quad \text{as} \ \ n \to \infty \tag{244}$$

and

$$\left\|T[\psi_n]' - T[\tilde{\sigma}]'\right\|_\infty \to 0 \quad \text{as} \ \ n \to \infty, \tag{245}$$

where $T[f]'$ denotes the derivative of the function $T[f](x)$ with respect to x. From (242), (243) and the definition (74) of the operator S we conclude that

$$\|S[T[\psi_n]] - S[T[\tilde{\sigma}]]\|_\infty \to 0 \ \text{ as } \ n \to \infty. \tag{246}$$

We rearrange (229) as

$$\psi_n = S[T[\psi_n]] + p + \tilde{\nu} + (S[T[\tilde{\sigma}]] - S[T[\psi_n]]) + (\psi_n - \tilde{\sigma}) \tag{247}$$

and define τ_n for $n = 1, 2, \ldots$ via the formula

$$\tau_n = \tilde{\nu} + (S[T[\tilde{\sigma}]] - S[T[\psi_n]]) + (\psi_n - \tilde{\sigma}). \tag{248}$$

From (240), (246), and (232) we conclude that

$$\|\tau_n\|_\infty \leqslant \frac{12\Gamma}{a} \exp(-a\lambda) \tag{249}$$

for all sufficiently large positive integers n. Moreover, it follows from (235) and (230) that

$$|\psi_n(\xi)| \leqslant 2\Gamma \exp(-a|\xi|) \quad \text{for all} \quad |\xi| \leqslant \sqrt{2}\lambda \tag{250}$$

whenever n is sufficiently large. By letting $\sigma = \psi_n$ and $\nu = \tau_n$ for some positive integer n which is sufficiently large for both (249) and (250) to hold, we obtain the conclusions of Theorem 7. □

9. Numerical experiments

In this section, we describe numerical experiments which, *inter alia*, illustrate one of the important consequences of the existence of nonoscillatory phase functions. Namely, that a large class of special functions can be evaluated to high accuracy using a number of operations which does not grow with order.

Although the proof of Theorem 7 suggests a numerical procedure for the construction of nonoscillatory phase functions, we utilize a different procedure here. It has the advantage that the coefficient q in the ordinary differential equation (1) need not be extended outside of the interval on which the nonoscillatory phase function is constructed. A paper describing this work is in preparation.

The code we used for these calculations was written in Fortran and compiled with the Intel Fortran Compiler version 12.1.3. All calculations were carried out in double precision arithmetic on a desktop computer equipped with an Intel Xeon X5690 CPU running at 3.47 GHz.

9.1. A nonoscillatory solution of the logarithm form of Kummer's equation.

In this experiment, we illustrate Theorem 7 in Section 4. We first construct a nonoscillatory solution r of the logarithm form of Kummer's equation

$$r''(t) - \frac{1}{4}\left(r'(t)\right)^2 + 4\lambda^2\left(\exp(r(t)) - q(t)\right) = 0 \tag{251}$$

on the interval $[-1,1]$, where λ = 1,000 and q is the function $[-1,1] \to \mathbb{R}$ defined by the formula

$$q(t) = \left(3 + \frac{1}{1+10t^2} + t^3\cos(5t)\right). \tag{252}$$

Then we compute the 500 leading Chebyshev coefficients of q and r.

We display the results of this experiment in Figures 1 and 2. Figure 1 contains plots of the functions q and r, while Figure 2 contains a plot of the base-10 logarithms of the absolute values of the leading Chebyshev coefficients of q and r.

We observe that, consistent with Theorem 7, the Chebyshev coefficients of both r and q decay exponentially, although those of r decay at a slightly slower rate.

9.2. Evaluation of Legendre polynomials.

In this experiment, we compare the cost of evaluating Legendre polynomials of large order using the standard recurrence relation with the cost of doing so with a nonoscillatory phase function.

For any integer $n \geqslant 0$, the Legendre polynomial $P_n(x)$ of order n is a solution of the second order differential equation

$$(1-t^2)y''(t) - 2ty'(t) + n(n+1)y(t) = 0. \tag{253}$$

Equation (253) can be put into the standard form

$$\psi''(t) + \left(\frac{1+n-nt^2-n^2(t^2-1)}{(1-t^2)^2}\right)\psi(t) = 0 \tag{254}$$

by introducing the transformation

$$\psi(t) = \sqrt{1-t^2}\,y(t). \tag{255}$$

Legendre polynomials satisfy the well-known three term recurrence relation

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t). \tag{256}$$

See, for instance, [11] for a discussion of the these and other properties of Legendre polynomials.

For each of 9 values of n, we proceed as follows. We sample 1000 random points

$$t_1, t_2, \ldots, t_{1000} \tag{257}$$

from the uniform distribution on the interval $(-1,1)$. Then we evaluate the Legendre polynomial of order n using the recurrence relation (256) at each of the points $t_1, t_2, \ldots, t_{1000}$. Next, we construct a nonoscillatory phase function for the ordinary differential equation (254) and use it evaluate the Legendre polynomial of order n at each of the points $t_1, t_2, \ldots, t_{1000}$. Finally, for each integer $j = 1, \ldots, 1000$, we compute the error in the approximation of $P_n(t_j)$ obtained from the nonoscillatory phase function by comparing it to the value obtained using the recurrence relation (we regard the recurrence relation as giving the more accurate approximation).

The results of this experiment are shown in Table 1. There, each row correponds to value of n. That value is listed n, as is the time required to compute each phase function for that value of n, the average time required to evaluate the Legendre polynomial of order n using the recurrence relation, the average cost of evaluating the Legendre polynomial of order n with the nonoscillatory phase function, and the largest of the absolute errors in the approximations of the quantities

$$P_n(t_1), P_n(t_2), \ldots, P_n(t_{1000})$$

obtained via the phase function method.

This experiment reveals that, as expected, the cost of evaluating $P_n(t)$ using the recurrence relation (256) grows as $O(n)$ while the cost of doing so with nonoscillatory phase function is independet of order.

However, it also exposes a limitation of phase functions. The values of $P_n(t)$ are obtained in part by evaluating sine and cosine of a phase function whose magnitude is on the order of n. This imposes limitations on the accuracy of the method due to the well-known difficulties in evaluating periodic functions of large arguments.

Figure 3 contains a plot of the nonoscillatory phase function for the equation (254) when n =1,000,000.

9.3. Evaluation of Bessel functions.

In this experiment, we compare the cost of evaluating Bessel functions of integer order via the standard recurrence relation with that of doing so using a nonoscillatory phase function.

We will denote by J_ν the Bessel function of the first kind of order ν. It is a solution of the second order differential equation

$$t^2 y''(t) + t y'(t) + (t^2 - \nu^2) y(t) = 0, \tag{258}$$

which can be brought into the standard form

$$\psi''(t) + \left(1 - \frac{\lambda^2 - 1/4}{t^2}\right)\psi(t) = 0 \tag{259}$$

via the transformation

$$\psi(t) = \sqrt{t}\, y(t). \tag{260}$$

An inspection of (259) reveals that J_ν is nonoscillatory on the interval

$$\left(0, \frac{1}{2}\sqrt{4\nu^2 - 1}\right) \tag{261}$$

and oscillatory on the interval

$$\left(\frac{1}{2}\sqrt{4\nu^2 - 1}, \infty\right). \tag{262}$$

The Bessel functions satisfy the three-term recurrence relation

$$J_{\nu+1}(t) = \frac{2\nu}{t} J_\nu(t) - J_{\nu-1}(t). \tag{263}$$

The recurrence (263) is numerically unstable in the forward direction; however, when evaluated in the direction of decreasing index, it yields a stable mechanism for evaluating Bessel functions of integer order (see, for instance, Chapter 3 of [11]).

For each of 9 values of n, we proceed as follows. First, we sample 1000 random points

$$t_1, t_2, \ldots, t_{1000} \tag{264}$$

from the uniform distribution on the interval $[2n, 3n]$. We then use the recurrence relation (263) to evaluate the Bessel function J_n of order n at the points $t_1, t_2, \ldots, t_{1000}$. Next, we construct a nonoscillatory phase function for the equation (260) on the interval $[2n, 3n]$ and use it to evaluate J_n at the points

$t_1, t_2, \ldots, t_{1000}$. Finally, for each integer $j = 1, \ldots, 1000$, we compute the error in the approximation of $J_n(t_j)$ obtained from the nonoscillatory phase function by comparing it to the value obtained using the recurrence relation (once again we regard the recurrence relation as giving the more accurate approximation).

The results of this experiment are displayed in Table 2. There, each row corresponds to one value of n. In addition to that value of n, it lists the time required to compute the phase function at order n, the average cost of evaluating J_n using the recurrence relation, the average cost of evaluating it with the nonoscillatory phase function, and the largest of the absolute errors in the approximations of the quantities

$$J_n(t_1), J_n(t_2), \ldots, J_n(t_{1000})$$

obtained via the phase function method.

We observe that while the cost of evaluating J_n using the recurrence relation (263) grows as $O(n)$, the time taken by the nonoscillatory phase function approach scales as $O(1)$. We also note that, as in the case of Legendre polynomials, there is some loss of accuracy with the phase function method due to the difficulties of evaluating trigonometric functions of large arguments.

10. Conclusions

We have shown that the solutions of a large class of second order differential equations can be accurately represented using nonoscillatory phase functions.

We have also presented the results of numerical experiments which demonstrate one of the applications of nonoscillatory phase functions: the evaluation of special functions at a cost which is independent of order. An efficient algorithm for the evaluation of highly oscillatory special functions will be reported at a later date. Other applications of this work include the computation of the zeros of special functions and the fast application of special function transforms. These topics will also be addressed by the authors at a later date.

11. Acknowledgements

Zhu Heitman was supported in part by the Office of Naval Research under contracts ONR N00014-10-1-0570 and ONR N00014-11-1-0718. James Bremer was supported in part by a fellowship from the Alfred P. Sloan Foundation and by National Science Foundation grant DMS-1418723. Vladimir Rokhlin was supported in part by Office of Naval Research contracts ONR N00014-10-1-0570 and ONR N00014-11-1-0718, and by the Air Force Office of Scientific Research under contract AFOSR FA9550-09-1-0241.

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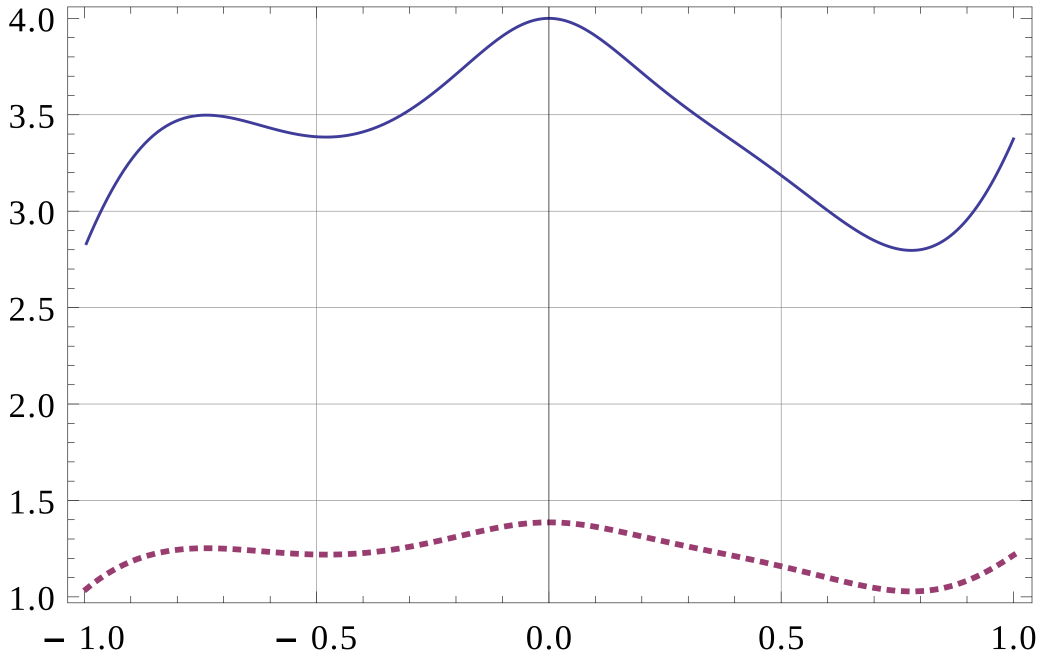

Figure 1: The function q defined by formula (252) in Section 9.1 (solid line) and the corresponding solution r of the logarithm form of Kummer's equation (52) when $\lambda = 1{,}000$ (dotted line).

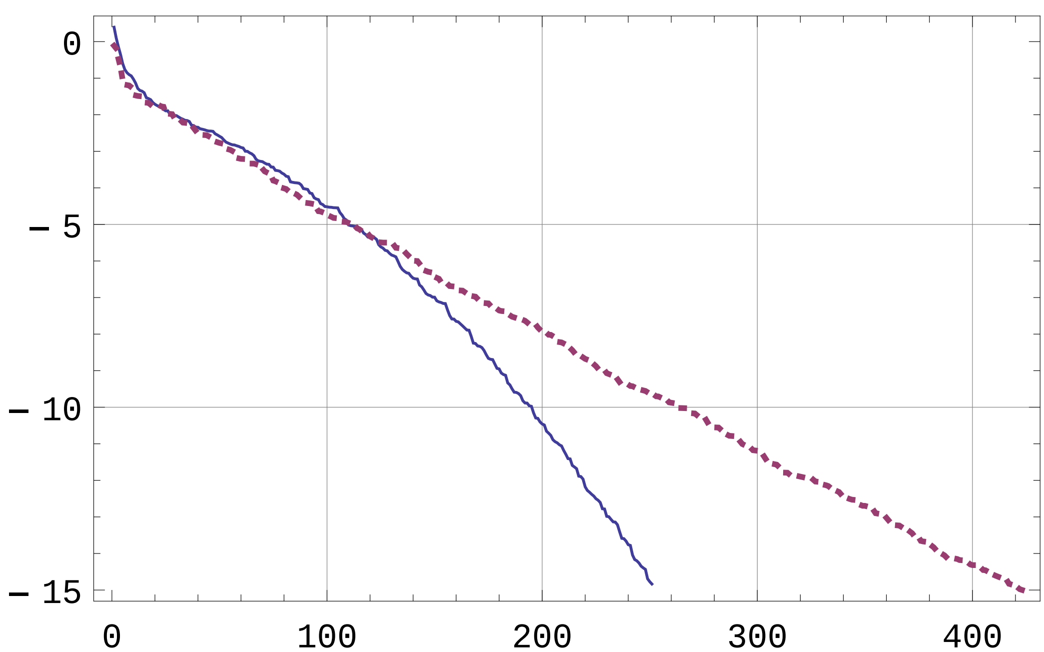

Figure 2: The base-10 logarithms of the leading Chebyshev coefficients of the function q defined by formula (252) in Section 9.1 (solid line) and of the associated nonoscillatory solution r of equation the logarithm form of Kummer's equation (52) when λ = 1,000 (dotted line).

n	Phase function construction time	Avg. phase function evaluation time	Avg. recurrence evaluation time	Largest absolute error
10^1	1.55×10^{-1} secs	1.29×10^{-6} secs	5.82×10^{-8} secs	5.16×10^{-14}
10^2	1.76×10^{-1} secs	1.29×10^{-6} secs	9.73×10^{-7} secs	1.59×10^{-13}
10^3	1.57×10^{-1} secs	1.29×10^{-6} secs	1.03×10^{-5} secs	6.13×10^{-13}
10^4	1.55×10^{-1} secs	1.29×10^{-6} secs	1.04×10^{-4} secs	1.20×10^{-12}
10^5	1.56×10^{-1} secs	1.31×10^{-6} secs	1.04×10^{-3} secs	9.79×10^{-12}
10^6	1.58×10^{-1} secs	1.40×10^{-6} secs	9.81×10^{-3} secs	2.40×10^{-11}
10^7	1.65×10^{-1} secs	1.40×10^{-6} secs	9.69×10^{-2} secs	8.59×10^{-11}
10^8	1.87×10^{-1} secs	1.42×10^{-6} secs	9.68×10^{-1} secs	1.71×10^{-10}
10^9	2.05×10^{-1} secs	1.34×10^{-6} secs	9.68×10^{-0} secs	6.11×10^{-10}

Table 1: **The evaluation of Legendre polynomials**. A comparison of the time required to evaluate the Legendre polynomial of order n using the standard recurrence relation and the time necessary to evaluate it using a nonoscillatory phase function. The recurrence relation approach scales as $O(n)$ while the phase function approach scales as $O(1)$.

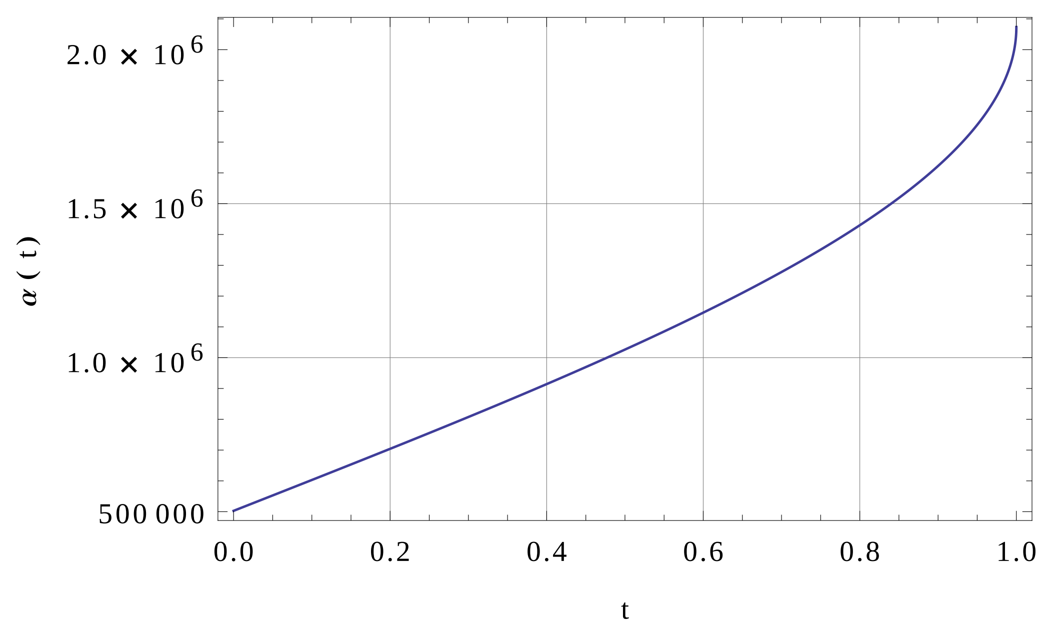

Figure 3: **A phase function for Legendre's differential equation**. A plot of the nonoscillatory phase function associated with Legendre's equation (253) at order $n = 1{,}000{,}000$. It is sufficient to construct the phase function on the interval $[0, 1)$ due to the symmetry properties of Legendre's differential equation.

n	Phase function construction time	Avg. phase function evaluation time	Avg. recurrence evaluation time	Largest absolute error
10^1	5.23×10^{-1} secs	1.30×10^{-6} secs	1.99×10^{-6} secs	2.81×10^{-14}
10^2	5.39×10^{-1} secs	1.31×10^{-6} secs	7.29×10^{-6} secs	7.85×10^{-14}
10^3	5.36×10^{-1} secs	1.37×10^{-6} secs	4.87×10^{-5} secs	2.40×10^{-13}
10^4	5.52×10^{-1} secs	1.33×10^{-6} secs	4.35×10^{-4} secs	1.01×10^{-12}
10^5	5.46×10^{-1} secs	1.49×10^{-6} secs	4.11×10^{-3} secs	3.18×10^{-12}
10^6	5.81×10^{-1} secs	1.44×10^{-6} secs	4.24×10^{-2} secs	8.57×10^{-12}
10^7	6.41×10^{-1} secs	1.45×10^{-6} secs	4.36×10^{-1} secs	5.98×10^{-11}
10^8	7.00×10^{-1} secs	1.35×10^{-6} secs	$4.39\times10^{+0}$ secs	1.14×10^{-10}
10^9	$1.26\times10^{+0}$ secs	1.41×10^{-6} secs	$4.42\times10^{+1}$ secs	2.43×10^{-10}

Table 2: **The evaluation of Bessel functions**. A comparison of the time required to evaluate the Bessel function J_n using the standard recurrence relation with that required to evaluate it using a nonoscillatory phase function. All of the points at which J_n was evaluated were in the interval $[2n, 3n]$. The recurrence relation approach scales as $O(n)$ in the order n while the time required by the phase function method is $O(1)$.