THE UNIVERSITY OF TORONTO

UNDERGRADUATE MATHEMATICS COMPETITION

Sunday, March 18, 2001

Time: 3 hours

1. Let $a, b, c > 0$, $a < bc$ and $1 + a^3 = b^3 + c^3$. Prove that $1 + a < b + c$.

2. Let $O = (0, 0)$ and $Q = (1, 0)$. Find the point P on the line with equation $y = x + 1$ for which the angle OPQ is a maximum.

3. (a) Consider the infinite integer lattice in the plane (*i.e.*, the set of points with integer coordinates) as a graph, with the edges being the lines of unit length connecting nearby points. What is the minimum number of colours that can be used to colour all the vertices and edges of this graph, so that

   (i) each pair of adjacent vertices gets two distinct colours;

   (ii) each pair of edges that meet at a vertex gets two distinct colours; and

   (iii) an edge is coloured differently than either of the two vertices at the ends?

   (b) Extend this result to lattices in real $n-$dimensional space.

4. Let $\mathbf{V}$ be the vector space of all continuous real-valued functions defined on the open interval $(-\pi/2, \pi/2)$, with the sum of two functions and the product of a function and a real scalar defined in the usual way.

   (a) Prove that the set $\{\sin x, \cos x, \tan x, \sec x\}$ is linearly independent.

   (b) Let $\mathbf{W}$ be the linear space generated by the four trigonometric functions given in (a), and let T be the linear transformation determined on $\mathbf{W}$ into $\mathbf{V}$ by $T(\sin x) = \sin^2 x$, $T(\cos x) = \cos^2 x$, $T(\tan x) = \tan^2 x$ and $T(\sec x) = \sec^2 x$. Determine a basis for the kernel of T.

   Notes. A subset $\{v_1, v_2, \cdots, v_k\}$ of a vertor space is *linearly independent* iff $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$ for scalars c_i implies that $c_1 = c_2 = \cdots = c_k = 0$. The *kernel* of a linear transformation is the subspace that T maps to the zero vector. A *basis* for a vector space is a linearly independent set of vectors for which every element of the space is some linear combination.

5. Let n be a positive integer and x a real number not equal to a nonnegative integer. Prove that
$$\frac{n}{x} + \frac{n(n-1)}{x(x-1)} + \frac{n(n-1)(n-2)}{x(x-1)(x-2)} + \cdots + \frac{n(n-1)(n-2)\cdots 1}{x(x-1)(x-2)\cdots(x-n+1)} = \frac{n}{x-n+1} .$$
   [This was a problem given by Samuel Beatty on a regular problem assignment to first year honours mathematics students in the 1930s.]

6. Prove that, for each positive integer n, the series
$$\sum_{k=1}^{\infty} \frac{k^n}{2^k}$$
   converges to twice an odd integer not less than $(n + 1)!$.

7. Suppose that $x \geq 1$ and that $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x and the fractional part $\{x\}$ satisfies $0 \leq x < 1$. Define
$$f(x) = \frac{\sqrt{\lfloor x \rfloor} + \sqrt{\{x\}}}{\sqrt{x}} .$$

(a) Determine the supremum, *i.e.*, the least upper bound, of the values of $f(x)$ for $1 \le x$.

(b) Let $x_0 \ge 1$ be given, and for $n \ge 1$, define $x_n = f(x_{n-1})$. Prove that $\lim_{n\to\infty} x_n$ exists.

8. A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let a be the length of a side, b be the length of a shorter diagonal and c be the length of a longer diagonal of a regular heptagon (so that $a < b < c$). Prove ONE of the following relationships:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

or

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 \ .$$

Saturday, March 9, 2002

Time: $3\frac{1}{2}$ hours

1. Let A, B, C be three pairwise orthogonal faces of a tetrahedran meeting at one of its vertices and having respective areas a, b, c. Let the face D opposite this vertex have area d. Prove that

$$d^2 = a^2 + b^2 + c^2 \ .$$

2. Angus likes to go to the movies. On Monday, standing in line, he noted that the fraction x of the line was in front of him, while $1/n$ of the line was behind him. On Tuesday, the same fraction x of the line was in front of him, while $1/(n+1)$ of the line was behind him. On Wednesday, the same fraction x of the line was in front of him, while $1/(n+2)$ of the line was behind him. Determine a value of n for which this is possible.

3. In how many ways can the rational 2002/2001 be written as the product of two rationals of the form $(n+1)/n$, where n is a positive integer?

4. Consider the parabola of equation $y = x^2$. The normal is constructed at a variable point P and meets the parabola again in Q. Determine the location of P for which the arc length along the parabola between P and Q is minimized.

5. Let n be a positive integer. Suppose that f is a function defined and continuous on $[0,1]$ that is differentiable on $(0,1)$ and satisfies $f(0) = 0$ and $f(1) = 1$. Prove that, there exist n [distinct] numbers x_i $(1 \le i \le n)$ in $(0,1)$ for which

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n \ .$$

6. Let $x, y > 0$ be such that $x^3 + y^3 \le x - y$. Prove that $x^2 + y^2 \le 1$.

7. Prove that no vector space over R is a finite union of proper subspaces.

8. (a) Suppose that P is an $n \times n$ nonsingular matrix and that u and v are column vectors with n components. The matrix $v^T P^{-1} u$ is 1×1, and so can be identified with a scalar. Suppose that its value is not equal to -1. Prove that the matrix $P + uv^T$ is nonsingular and that

$$(P + uv^T)^{-1} = P^{-1} - \frac{1}{\alpha} P^{-1} u v^T P^{-1}$$

where v^T denotes the transpose of v and $\alpha = 1 + v^T P^{-1} u$.

(b) Explain the situation when $\alpha = 0$.

9. A sequence whose entries are 0 and 1 has the property that, if each 0 is replaced by 01 and each 1 by 001, then the sequence remains unchanged. Thus, it starts out as $010010101001 \cdots$. What is the 2002th term of the sequence?

Sunday, March 16, 2003

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

1. Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{n^2}\right) .$$

[$\tan^{-1}$ denotes the (composition) inverse function for tan.]

2. Let a, b, c be positive real numbers for which $a + b + c = abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2} .$$

3. Solve the differential equation

$$y'' = yy' .$$

4. Show that n divides the integer nearest to

$$\frac{(n+1)!}{e} .$$

5. For $x > 0$, $y > 0$, let $g(x, y)$ denote the minimum of the three quantities, x, $y + 1/x$ and $1/y$. Determine the maximum value of $g(x, y)$ and where this maximum is assumed.

6. A set of n lighbulbs, each with an *on-off* switch, numbered $1, 2, \cdots, n$ are arranged in a line. All are initially off. Switch 1 can be operated at any time to turn its bulb on of off. Switch 2 can turn bulb 2 on or off if and only if bulb 1 is off; otherwise, it does not function. For $k \geq 3$, switch k can turn bulb k on or off if and only if bulb $k-1$ is off and bulbs $1, 2, \cdots, k-2$ are all on; otherwise it does not function.

(a) Prove that there is an algorithm that will turn all of the bulbs on.

(b) If x_n is the length of the shortest algorithm that will turn on all n bulbs when they are initially off, determine the largest prime divisor of $3x_n + 1$ when n is odd.

7. Suppose that the polynomial $f(x)$ of degree $n \geq 1$ has all real roots and that $\lambda > 0$. Prove that the set $\{x \in \mathbb{R} : |f(x)| \leq \lambda |f'(x)|\}$ is a finite union of closed intervals whose total length is equal to $2n\lambda$.

8. Three matrices A, B and $A + B$ have rank 1. Prove that either all the rows of A and B are multiples of one and the same vector, or that all of the columns of A and B are multiples of one and the same vector.

9. Prove that the integral

$$\int_0^\infty \frac{\sin^2 x}{\pi^2 - x^2} dx$$

exists and evaluate it.

10. Let G be a finite group of order n. Show that n is odd if and only if each element of G is a square.

Sunday, March 14, 2004

Time: $3\frac{1}{2}$ hours

1. Prove that, for any complex numbers z and w,

$$(|z| + |w|)\left|\frac{z}{|z|} + \frac{w}{|w|}\right| \le 2|z + w| \ .$$

2. Prove that

$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} + \cdots \ .$$

3. Suppose that u and v are positive integer divisors of the positive integer n and that $uv < n$. Is it necessarily so that the greatest common divisor of n/u and n/v exceeds 1?

4. Let n be a positive integer exceeding 1. How many permutations $\{a_1, a_2, \cdots, a_n\}$ of $\{1, 2, \cdots, n\}$ are there which maximize the value of the sum

$$|a_2 - a_1| + |a_3 - a_2| + \cdots + |a_{i+1} - a_i| + \cdots + |a_n - a_{n-1}|$$

over all permutations? What is the value of this maximum sum?

5. Let A be a $n \times n$ matrix with determinant equal to 1. Let B be the matrix obtained by adding 1 to every entry of A. Prove that the determinant of B is equal to $1 + s$, where s is the sum of the n^2 entries of A^{-1}.

6. Determine

$$\left(\int_0^1 \frac{dt}{\sqrt{1-t^4}}\right) \div \left(\int_0^1 \frac{dt}{\sqrt{1+t^4}}\right) .$$

7. Let a be a parameter. Define the sequence $\{f_n(x) : n = 0, 1, 2, \cdots\}$ of polynomials by

$$f_0(x) \equiv 1$$

$$f_{n+1}(x) = xf_n(x) + f_n(ax)$$

for $n \ge 0$.

(a) Prove that, for all n, x,

$$f_n(x) = x^n f_n(1/x) \ .$$

(b) Determine a formula for the coefficient of x^k $(0 \le k \le n)$ in $f_n(x)$.

8. Let V be a complex $n-$dimensional inner product space. Prove that

$$|u|^2|v|^2 - \frac{1}{4}|u - v|^2|u + v|^2 \le |(u, v)|^2 \le |u|^2|v|^2 \ .$$

9. Let $ABCD$ be a convex quadrilateral for which all sides and diagonals have rational length and AC and BD intersect at P. Prove that AP, BP, CP, DP all have rational length.

Saturday, March 12, 2005

Time: $3\frac{1}{2}$ hours

1. Show that, if $-\pi/2 < \theta < \pi/2$, then

$$\int_0^\theta \log(1 + \tan\theta \tan x)dx = \theta \log \sec \theta \ .$$

2. Suppose that f is continuously differentiable on $[0,1]$ and that $\int_0^1 f(x)dx = 0$. Prove that

$$2\int_0^1 f(x)^2 dx \leq \int_0^1 |f'(x)|dx \cdot \int_0^1 |f(x)|dx \ .$$

3. How many $n \times n$ invertible matrices A are there for which all the entries of both A and A^{-1} are either 0 or 1?

4. Let a be a nonzero real and $\mathbf{u}$ and $\mathbf{v}$ be real 3-vectors. Solve the equation

$$2a\mathbf{x} + (\mathbf{v} \times \mathbf{x}) + \mathbf{u} = \mathbf{O}$$

for the vector $\mathbf{x}$.

5. Let $f(x)$ be a polynomial with real coefficients, evenly many of which are nonzero, which is *palindromic*. This means that the coefficients read the same in either direction, *i.e.* $a_k = a_{n-k}$ if $f(x) = \sum_{k=0}^n a_k x^k$ or $f(x) = x^n f(1/x)$, where n is the degree of the polynomial. Prove that $f(x)$ has at least one root of absolute value 1.

6. Let G be a subgroup of index 2 contained in S_n, the group of all permutations of n elements. Prove that $G = A_n$, the alternating group of all even permutations.

7. Let $f(x)$ be a nonconstant polynomial that takes only integer values when x is an integer, and let P be the set of all primes that divide $f(m)$ for at least one integer m. Prove that P is an infinite set.

8. Let $AX = B$ represent a system of m linear equations in n unknowns, where $A = (a_{ij})$ is an $m \times n$ matrix, $X = (x_1, \cdots, x_n)^{\mathbf{t}}$ is an $n \times 1$ vector and $B = (b_1, \cdots, b_m)^{\mathbf{t}}$ is an $m \times 1$ vector. Suppose that there exists at least one solution for $AX = B$. Given $1 \leq j \leq n$, prove that the value of the jth component is the same for every solution X of $AX = B$ if and only if the rank of A is decreased if the jth column of A is removed.

9. Let S be the set of all real-valued functions that are defined, positive and twice continuously differentiable on a neighbourhood of 0. Suppose that a and b are real parameters with $ab \neq 0$, $b < 0$. Define operators from S to $\mathbf{R}$ as follows:

$$A(f) = f(0) + af'(0) + bf''(0) \ ;$$

$$G(f) = \exp A(\log f) \ .$$

(a) Prove that $A(f) \leq G(f)$ for $f \in S$;

(b) Prove that $G(f+g) \leq G(f) + G(g)$ for $f, g \in S$;

(c) Suppose that H is the set of functions in S for which $G(f) \leq f(0)$. Give examples of nonconstant functions, one in H and one not in H. Prove that, if $\lambda > 0$ and $f, g \in H$, then λf, $f + g$ and fg all belong to H.

10. Let n be a positive integer exceeding 1. Prove that, if a graph with $2n+1$ vertices has at least $3n+1$ edges, then the graph contains a circuit (*i.e.*, a closed non-self-intersecting chain of edges whose terminal point is its initial point) with an even number of edges. Prove that this statement does not hold if the number of edges is only $3n$.

Sunday, March 12, 2006

Time: $3\frac{1}{2}$ hours

1. (a) Suppose that a 6×6 square grid of unit squares (chessboard) is tiled by 1×2 rectangles (dominoes). Prove that it can be decomposed into two rectangles, tiled by disjoint subsets of the dominoes.

   (b) Is the same thing true for an 8×8 array?

2. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^3$ and define the operator P by $P(\mathbf{x}) = \mathbf{u} \times \mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^3$ (where $\times$ denotes the cross product).

   (a) Describe the operator $I + P^2$.

   (b) Describe the action of the operator $I + (\sin\theta)P + (1 - \cos\theta)P^2$.

3. Let $p(x)$ be a polynomials of positive degree n with n distinct real roots $a_1 < a_2 < \cdots < a_n$. Let b be a real number for which $2b < a_1 + a_2$. Prove that

$$2^{n-1}|p(b)| \geq |p'(a_1)(b - a_1)| \ .$$

4. Two parabolas have parallel axes and intersect in two points. Prove that their common chord bisects the segments whose endpoints are the points of contact of their common tangent.

5. Suppose that you have a 3×3 grid of squares. A *line* is a set of three squares in the same row, the same column or the same diagonal; thus, there are eight lines.

   Two players A and B play a game. They take alternate turns, A putting a 0 in any unoccupied square of the grid and B putting a 1. The first player is A, and the game cannot go on for more than nine moves. (The play is similar to noughts-and-crosses, or tictactoe.) A move is *legitimate* if it does not result in two lines of squares being filled in with different sums. The winner is the last player to make a legitimate move.

   (For example, if there are three 0s down the diagonal, then B can place a 1 in any vacant square provided it completes no other line, for then the sum would differ from the diagonal sum. If there are two zeros at the top of the main diagonal and two ones at the left of the bottom line, then the lower right square cannot be filled by either player, as it would result in two lines with different sums.)

   (a) What is the maximum number of legitimate moves possible in a game?

   (b) What is the minimum number of legitimate moves possible in a game that would not leave a legitimate move available for the next player?

   (c) Which player has a winning strategy? Explain.

6. Suppose that k is a positive integer and that

$$f(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \cdots + a_k e^{\lambda_k t}$$

   where $a_1, \cdots, a_k, \lambda_1, \cdots, \lambda_k$ are real numbers with $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. Prove that $f(t) = 0$ has finitely many real solutions. What is the maximum number of solutions possible, as a function of k?

7. Let A be a real 3×3 invertible matrix for which the sums of the rows, columns and two diagonals are all equal. Prove that the rows, columns and diagonal sums of A^{-1} are all equal.

8. Let $f(x)$ be a real function defined and twice differentiable on an open interval containing $[-1, 1]$. Suppose that $0 < \alpha \leq \gamma$ and that $|f(x)| \leq \alpha$ and $|f''(x)| \leq \gamma$ for $-1 \leq x \leq 1$. Prove that

$$|f'(x)| \leq 2\sqrt{\alpha\gamma}$$

for $-1 \leq x \leq 1$. (Part marks are possible for the weaker inequality $|f'(x)| \leq \alpha + \gamma$.)

9. A high school student asked to solve the surd equation

$$\sqrt{3x-2} - \sqrt{2x-3} = 1$$

gave the following answer: *Squaring both sides leads to*

$$3x - 2 - 2x - 3 = 1$$

so $x = 6$. The answer is, in fact, correct.

Show that there are infinitely many real quadruples (a, b, c, d) for which this method leads to a correct solution of the surd equation

$$\sqrt{ax-b} - \sqrt{cx-d} = 1 \ .$$

10. Let P be a planar polygon that is not convex. The vertices can be classified as either *convex* or *concave* according as to whether the angle at the vertex is less than or greater than 180° respectively. There must be at least two convex vertices. Select two consecutive convex vertices (*i.e.*, two interior angles less than 180° for which all interior angles in between exceed 180°) and join them by a segment. Reflect the edges between these two convex angles in the segment to form along with the other edges of P a polygon P_1. If P_1 is not convex, repeat the process, reflecting some of the edges of P_1 in a segment joining two consecutive convex vertices, to form a polygon P_2. Repeat the process. Prove that, after a finite number of steps, we arrive at a polygon P_n that is convex.

Sunday, March 11, 2007

Time: $3\frac{1}{2}$ hours

1. A $m \times n$ rectangular array of distinct real numbers has the property that the numbers in each row increase from left to right. The entries in each column, individually, are rearranged so that the numbers in each column increase from top to bottom. Prove that it the final array, the numbers in each row will increase from left to right.

2. Determine distinct positive integers a, b, c, d, e such that the five numbers a, b^2, c^3, d^4, e^5 constitute an arithmetic progression. (The difference between adjacent pairs is the same.)

3. Prove that the set $\{1, 2, \cdots, n\}$ can be partitioned into k subsets with the same sum if and only if k divides $\frac{1}{2}n(n+1)$ and $n \geq 2k - 1$.

4. Suppose that $f(x)$ is a continuous real-valued function defined on the interval $[0, 1]$ that is twice differentiable on $(0, 1)$ and satisfies (i) $f(0) = 0$ and (ii) $f''(x) > 0$ for $0 < x < 1$.

   (a) Prove that there exists a number a for which $0 < a < 1$ and $f'(a) < f(1)$;

   (b) Prove that there exists a unique number b for which $a < b < 1$ and $f'(a) = f(b)/b$.

5. For $x \leq 1$ and $x \neq 0$, let

$$f(x) = \frac{-8[1-(1-x)^{1/2}]^3}{x^2} \ .$$

(a) Prove that $\lim_{x \to 0} f(x)$ exists. Take this as the value of $f(0)$.

(b) Determine the smallest closed interval that contains the set of all values assumed by $f(x)$ on its domain.

(c) Prove that $f(f(f(x))) = f(x)$ for all $x \leq 1$.

6. Let $h(n)$ denote the number of finite sequences $\{a_1, a_2, \cdots, a_k\}$ of positive integers exceeding 1 for which $k \geq 1$, $a_1 \geq a_2 \geq \cdots \geq a_k$ and $n = a_1 a_2 \cdots a_k$. (For example, if $n = 20$, there are four such sequences $\{20\}$, $\{10, 2\}$, $\{5, 4\}$ and $\{5, 2, 2\}$ and $h(20) = 4$.

Prove that

$$\sum_{n=1}^{\infty} \frac{h(n)}{n^2} = 1 \ .$$

7. Find the Jordan canonical form of the matrix $\mathbf{u}\mathbf{v}^{\mathbf{t}}$ where $\mathbf{u}$ and $\mathbf{v}$ are column vectors in $\mathbf{C}^n$. (The superscript $\mathbf{t}$ denotes the transpose.)

8. Suppose that n points are given in the plane, not all collinear. Prove that there are at least n distinct straight lines that can be drawn through pairs of the points.

9. Which integers can be written in the form

$$\frac{(x+y+z)^2}{xyz}$$

where x, y, z are positive integers?

10. Solve the following differential equation

$$2y' = 3|y|^{1/3}$$

subject to the intial conditions

$$y(-2) = -1 \qquad \text{and} \qquad y(3) = 1 \ .$$

Your solution should be everywhere differentiable.

Sunday, March 9, 2008

Time: $3\frac{1}{2}$ hours

1. Three angles of a heptagon (7-sided polygon) inscribed in a circle are equal to 120°. Prove that at least two of its sides are equal.

2. (a) Determine a real-valued function g defined on the real numbers that is decreasing and for which $g(g(x)) = 2x + 2$.

(b) Prove that there is no real-valued function f defined on the real numbers that is decreasing and for which $f(f(x)) = x + 1$.

3. Suppose that a is a real number and the sequence $\{a_n\}$ is defined recursively by $a_0 = a$ and

$$a_{n+1} = a_n(a_n - 1)$$

for $n \geq 0$. Find the values of a for which the sequence $\{a_n\}$ converges.

4. Suppose that u, v, w, z are complex numbers for which $u + v + w + z = u^2 + v^2 + w^2 + z^2 = 0$. Prove that

$$(u^4 + v^4 + w^4 + z^4)^2 = 4(u^8 + v^8 + w^8 + z^8) \ .$$

5. Suppose that $a, b, c \in \mathbf{C}$ with $ab = 1$. Evaluate the determinant of

$$\begin{pmatrix} c & a & a^2 & \cdots & a^{n-1} \\ b & c & a & \cdots & a^{n-2} \\ b^2 & b & c & \cdots & a^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ b^{n-1} & b^{n-2} & & \cdots & c \end{pmatrix}$$

6. 2008 circular coins, possibly of different diameters, are placed on the surface of a flat table in such a way that no coin is on top of another coin. What is the largest number of points at which two of the coins could be touching?

7. Let G be a group of finite order and identity e. Suppose that ϕ is an automorphism of G onto itself with the following properties: (1) $\phi(x) = x$ if and only if $x = e$; (2) $\phi(\phi(x)) = x$ for each element x of G. (The mapping ϕ has the property that it is one-one onto and that $\phi(xy) = \phi(x)\phi(y)$ for each pair x, y of elements of G.)

   (a) Give an example of a group and automorphism for which these conditions are satisfied.

   (b) Prove that G is commutative (*i.e.*, $xy = yx$ for each pair x, y of elements in G).

8. Let $b \geq 2$ be an integer base of numeration and let $1 \leq r \leq b - 1$. Determine the sum of all $r-$digit numbers of the form

$$\overline{a_{r-1}a_{r-2}\cdots a_2a_1a_0} \equiv a_{r-1}b^{r-1} + a_{r-2}b^{r-2} + \cdots + a_1r + a_0$$

whose digits increase strictly from left to right: $1 \leq a_{r-1} < a_{r-2} < \cdots < a_1 < a_0 \leq b - 1$.

9. For each positive integer n, let

$$S(n) = \sum_{k=1}^{n} \frac{2^k}{k^2} \ .$$

Prove that $S(n+1)/S(n)$ is not a rational function of n. [A *rational function* is one that can be written as a ratio of two polynomials.]

10. A point is chosen at random (with the uniform distribution) on each side of a unit square. What is the probability that the four points are the vertices of a quadrilateral with area exceeding $\frac{1}{2}$?

Sunday, March 8, 2009

Time: $3\frac{1}{2}$ hours

1. Determine the supremum and the infimum of

$$\frac{(x-1)^{x-1}x^x}{(x-(1/2))^{2x-1}}$$

for $x > 1$.

2. Let n and k be integers with $n \geq 0$ and $k \geq 1$. Let $\mathbf{x}_0$, $\mathbf{x}_1$, $\cdots$, $\mathbf{x}_n$ be $n+1$ distinct points in $\mathbf{R}^k$ and let y_0, y_1, $\cdots$, y_n be $n+1$ real numbers (not necessarily distinct). Prove that there exists a polynomial p of degree at most n in the coordinates of $\mathbf{x}$ with respect to the standard basis for which $p(\mathbf{x}_i) = y_i$ for $0 \leq i \leq n$.

3. For each positive integer n, let $p(n)$ be the product of all positive integral divisors of n. Is it possible to find two distinct positive integers m and n for which $p(m) = p(n)$?

4. Let $\{a_n\}$ be a real sequence for which
$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$
converges. Prove that
$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 0 \; .$$

5. Find a 3×3 matrix A with elements in $\mathbb{Z}_2$ for which $A^7 = I$ and $A \neq I$. (Here, I is the identity matrix and $\mathbb{Z}_2$ is the field of two elements 0 and 1 where addition and multiplication are defined modulo 2.)

6. Determine all solutions in nonnegative integers (x, y, z, w) to the equation
$$2^x 3^y - 5^z 7^w = 1 \; .$$

7. Let $n \geq 2$. Minimize $a_1 + a_2 + \cdots + a_n$ subject to the constraints $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ and $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 1$. (When $n = 2$, the latter condition is $a_1a_2 = 1$; when $n \geq 3$, the sum on the left has exactly n terms.)

8. Let a, b, c be members of a real inner-product space $(V, \langle , \rangle)$ whose norm is given by $\|x\|^2 = \langle x, x\rangle$. (You may assume that V is $\mathbf{R}^n$ if you wish. Prove that
$$\|a + b\| + \|b + c\| + \|c + a\| \leq \|a\| + \|b\| + \|c\| + \|a + b + c\|$$
for $a, b, c, \in V$.

9. Let p be a prime congruent to 1 modulo 4. For each real number x, let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Determine
$$\sum \left\{ \left\{ \frac{k^2}{p} \right\} : 1 \leq k \leq \frac{1}{2}(p-1) \right\} .$$

10. Suppose that a path on a $m \times n$ grid consisting of the lattice points $\{(x, y) : 1 \leq x \leq m, 1 \leq y \leq n\}$ (x and y both integers) consisting of $mn - 1$ unit segments begins at the point $(1, 1)$, passes through each point of the grid exactly once, does not intersect itself and finishes at the point (m, n). Show that the path partitions the rectangle bounded by the lines $x = 1$, $x = m$, $y = 1$, $y = n$ into two subsets of equal area, the first consisting of regions opening to the left or up, and the second consisting of regions opening to the right or down.

Sunday, March 7, 2010

Time: $3\frac{1}{2}$ hours

vskip 0.5cm

1. Let F_1 and F_2 be the foci of an ellipse and P be a point in the plane of the ellipse. Suppose that G_1 and G_2 are points on the ellipse for which PG_1 and PG_2 are tangents to the ellipse. Prove that $\angle F_1PG_1 = \angle F_2PG_2$.

2. Let $u_0 = 1$, $u_1 = 2$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Prove that, for every nonnegative integer n,
$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} .$$

3. Let $\mathbf{a}$ and $\mathbf{b}$, the latter nonzero, be vectors in $\mathbb{R}^3$. Determine the value of λ for which the vector equation
$$\mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \lambda \mathbf{b}$$
is solvable, and then solve it.

4. The plane is partitioned into n regions by three families of parallel lines. What is the least number of lines to ensure that $n \geq 2010$?

5. Let m be a natural number, and let $c, a_1, a_2, \cdots, a_m$ be complex numbers for which $|a_i| = 1$ for $i = 1, 2, \cdots, m$. Suppose also that
$$\lim_{n \to \infty} \sum_{i=1}^{m} a_i^n = c .$$
Prove that $c = m$ and that $a_i = 1$ for $i = 1, 2, \cdots, m$.

6. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which
$$f(x)f(x+1) = g(h(x)) ,$$

7. Suppose that f is a continuous real-valued function defined on the closed interval $[0, 1]$ and that
$$\left(\int_0^1 xf(x)dx \right)^2 = \left(\int_0^1 f(x)dx \right) \left(\int_0^1 x^2 f(x)dx \right) .$$
Prove that there is a point $c \in (0, 1)$ for which $f(c) = 0$.

8. Let A be an invertible symmetric $n \times n$ matrix with coefficients $\{a_{i,j}\}$ in $\mathbb{Z}_2$. Prove that there is an $n \times n$ matrix with coefficients in $\mathbb{Z}_2$ such that $A = M^{\mathbf{t}} M$ only if $a_{i,i} \neq 0$ for some i.

   [$\mathbb{Z}_2$ refers to the field of integers modulo 2 with two elements $0, 1$ for which $1 + 1 = 0$. $M^{\mathbf{t}}$ refers to the transpose of the matrix M.]

9. Let f be a real-valued functions defined on $\mathbb{R}$ with a continuous third derivative, let $S_0 = \{x : f(x) = 0\}$, and, for $k = 1, 2, 3$, $S_k = \{x : f^{(k)}(x) = 0\}$, where $f^{(k)}$ denotes the kth derivative of f. Suppose also that $\mathbb{R} = S_0 \cup S_1 \cup S_2 \cup S_3$. Must f be a polynomial of degree not exceeding 2?

10. Prove that the set $\mathbb{Q}$ of rationals can be written as the union of countably many subsets of $\mathbb{Q}$ each of which is dense in the set $\mathbb{R}$ of real numbers.

Sunday, March 6, 2011

Time: $3\frac{1}{2}$ hours

1. Let S be a nonvoid set of real numbers with the property that, for each real number x, there is a unique real number $f(x)$ belonging to S that is farthest from x, *i.e.*, for each y in S distinct from $f(x)$, $|x - f(x)| > |x - y|$. Prove that S must be a singleton.

2. Let u and v be positive reals. Minimize the larger of the two values
$$2u + \frac{1}{v^2} \qquad \text{and} \qquad 2v + \frac{1}{u^2} .$$

3. Suppose that S is a set of n nonzero real numbers such that exactly p of them are positive and exactly q are negative. Determine all the pairs (n, p) such that exactly half of the threefold products abc of distinct elements a, b, c of S are positive.

4. Let $\{b_n : n \geq 1\}$ be a sequence of positive real numbers such that

$$3b_{n+2} \geq b_{n+1} + 2b_n$$

for every positive integer n. Prove that either the sequence converges or that it diverges to infinity.

5. Solve the system

$$x + xy + xyz = 12$$

$$y + yz + yzx = 21$$

$$z + zx + zxy = 30 \ .$$

6. Two competitors play badminton. They play two games, each winning one of them. They then play a third game to determine the overall winner of the match. The winner of a game of badminton is the first player to score at least 21 points with a lead of at least 2 points over the other player.

   In this particular match, it is observed that the scores of each player listed in order of the games form an arithmetic progression with a nonzero common difference. What are the scores of the two players in the third game?

7. Suppose that there are 2011 students in a school and that each students has a certain number of friends among his schoolmates. It is assumed that if A is a friend of B, then B is a friend of A, and also that there may exist certain pairs that are not friends. Prove that there is a nonvoid subset S of these students for which every student in the school has an even number of friends in S.

8. The set of transpositions of the symmetric group S_5 on $\{1, 2, 3, 4, 5\}$ is

$$\{(12), (13), (14), (15), (23), (24), (25), (34), (35), (45)\}$$

where (ab) denotes the permutation that interchanges a and b and leaves every other element fixed. Determine a product of all transpositions, each occuring exactly once, that is equal to the identity permutation ϵ, which leaves every element fixed.

9. Suppose that A and B are two square matrices of the same order for which the indicated inverses exist. Prove that

$$(A + AB^{-1}A)^{-1} + (A + B)^{-1} = A^{-1} \ .$$

10. Suppose that p is an odd prime. Determine the number of subsets S contained in $\{1, 2, \cdots, 2p - 1, 2p\}$ for which (a) S has exactly p elements, and (b) the sum of the elements of S is a multiple of p.

Saturday, March 10, 2012

Time: $3\frac{1}{2}$ hours

1. An equilateral triangle of side length 1 can be covered by five equilateral triangles of side length u. Prove that it can be covered by four equilateral triangles of side length u. (A triangle is a closed convex set that contains its three sides along with its interior.)

2. Suppose that f is a function defined on the set Z of integers that takes integer values and satisfies the condition that $f(b) - f(a)$ is a multiple of $b - a$ for every pair a, b, of integers. Suppose also that p is a polynomial with integer coefficients such that $p(n) = f(n)$ for infinitely many integers n. Prove that $p(x) = f(x)$ for every positive integer x.

3. Given the real numbers a, b, c not all zero, determine the real solutions x, y, z, u, v, w for the system of equations:

$$\begin{aligned} x^2 + v^2 + w^2 &= a^2 \\ u^2 + y^2 + w^2 &= b^2 \\ u^2 + v^2 + z^2 &= c^2 \\ u(y+z) + vw &= bc \\ v(x+z) + wu &= ca \\ w(x+y) + uv &= ab. \end{aligned}$$

4. (a) Let n and k be positive integers. Prove that the least common multiple of $\{n, n+1, n+2, \cdots, n+k\}$ is equal to

$$rn\binom{n+k}{k}$$

for some positive integer r.

(b) For each positive integer k, prove that there exist infinitely many positive integers n, for which the number r defined in part (a) is equal to 1.

5. Let $\mathfrak{C}$ be a circle and Q a point in the plane. Determine the locus of the centres of those circles that are tangent to $\mathfrak{C}$ and whose circumference passes through Q.

6. Find all continuous real-valued functions defined on R that satisfy $f(0) = 0$ and

$$f(x) - f(y) = (x - y)g(x + y)$$

for some real valued function $g(x)$.

7. Consider the following problem:

Suppose that $f(x)$ is a continuous real-valued function defined on the interval $[0, 2]$ for which

$$\int_0^2 f(x)dx = \int_0^2 (f(x))^2 dx \ .$$

Prove that there exists a number $c \in [0, 2]$ for which either $f(c) = 0$ or $f(c) = 1$.

(a) Criticize the following solution:

Solution. Clearly $\int_0^2 f(x)dx \geq 0$. By the extreme value theorem, there exist numbers u and v in $[0, 2]$ for which $f(u) \leq f(x) \leq f(v)$ for $0 \leq x \leq 2$. Hence

$$f(u)\int_0^2 f(x)dx \leq \int_0^2 f(x)^2 dx \leq f(v)\int_0^2 f(x)dx \ .$$

Since $\int_0^2 f(x)^2 dx = 1 \cdot \int_0^2 f(x)dx$, by the intermediate value theorem, there exists a number $c \in [0, 2]$ for which $f(c) = 1$. □

(b) Show that there is a nontrivial function f that satisfies the conditions of the problem but that never assumes the value 1.

(c) Provide a complete solution of the problem.

8. Determine the area of the set of points (x, y) in the plane that satisfy the two inequalities:

$$\begin{aligned} x^2 + y^2 &\leq 2 \\ x^4 + x^3y^3 &\leq xy + y^4. \end{aligned}$$

9. In a round-robin tournament of $n \geq 2$ teams, each pair of teams plays exactly one game that results in a win for one team and a loss for the other (there are no ties).

   (a) Prove that the teams can be labelled t_1, t_2, $\cdots$, t_n, so that, for each i with $1 \leq i \leq n-1$, team t_i beats t_{i+1}.

   (b) Suppose that a team t has the property that, for each other team u, one can find a chain u_1, u_2, $\cdots$, u_m of (possibly zero) distinct teams for which t beats u_1, u_i beats u_{i+1} for $1 \leq i \leq m-1$ and u_m beats u. Prove that *all* of the n teams can be ordered as in (a) so that $t = t_1$ and each t_i beats t_{i+1} for $1 \leq i \leq n-1$.

   (c) Let T denote the set of teams who can be labelled as t_1 in an ordering of teams as in (a). Prove that, in any ordering of teams as in (a), all the teams in T occur before all the teams that are not in T.

10. Let A be a square matrix whose entries are complex numbers. Prove that $A^* = A$ if and only if $AA^* = A^2$.

    Notes. For any $m \times n$ matrix M with entries m_{ij}, the *hermitian transpose* M^* is the $n \times m$ matrix M^* obtained by taking the complex conjugates of entries of M and transposing; thus, the (i, j)th element of M^* is $\overline{m_{ji}}$. In particular, for the complex column vector x with ith entry x_i, x^* is a row vector whose ith entry is $\bar{x}_i$. The inner product $\langle x, y \rangle$ of two column vectors is $\sum \bar{x}_i y_i = x^* y$, and we have that $\langle x, Ay \rangle = \langle A^* x, y \rangle$. A matrix for which $A^* = A$ is said to be *hermitian.*