

PUTNAM PROBLEMS
COMBINATORICS

2008-B-6. Let n and k be positive integers. Say that a permutation σ of $\{1, 2, \dots, n\}$ is k -limited if $|\sigma(i) - i| \leq k$ for all i . Prove that the number of k -limited permutations of $\{1, 2, \dots, n\}$ is odd if and only if $n \equiv 0$ or $1 \pmod{2k+1}$.

2007-B-6. For each positive integer n , let $f(n)$ be the number of ways to make $n!$ cents using an unordered collection of coins, each worth $k!$ cents for some k , $1 \leq k \leq n$. Prove that for some constant C , independent of n ,

$$n^{n^2/2 - Cn} e^{-n^2/4} \leq f(n) \leq n^{n^2/2 + Cn} e^{-n^2/4} .$$

2006-A-2. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

2006-A-4. Let $S = \{1, 2, \dots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

- (i) $\pi(k) > \pi(k+1)$ for $k = 1$;
- (ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;
- (iii) $\pi(k-1) < \pi(k)$ for $k = n$.

(For example, if $n = 5$ and π takes values at 1, 2, 3, 4, 5 of 2, 1, 4, 5, 3, then π has a local maximum of 2 at $k = 1$ and a local maximum of 5 at $k = 4$.)

What is the average number of local maxima of a permutation of S , averaging over all permutations of S ?

2005-A-2. Let $S = \{(a, b) : a = 1, 2, \dots, n, b = 1, 2, 3\}$. A *rook tour* of S is a polygonal path made up of line segments connecting points p_1, p_2, \dots, p_{3n} in sequence such that (i) $p_i \in S$, (ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$, (iii) for each $p \in S$, there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$.

(Here is an example of a rook tour for $n = 5$:

$$(1, 1), (2, 1), (2, 2), (2, 1), (3, 1), (3, 2), (3, 3), (3, 2) \\ (3, 1), (4, 1), (4, 2), (4, 3), (5, 3), (5, 2), (5, 1) .$$

2005-B-4. For positive integers m and n , let $f(m, n)$ denote the number of n -tuples (x_1, x_2, \dots, x_n) of integers such that $|x_1| + |x_2| + \dots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

2005-B-6. Let S_n denote the set of all permutations of the numbers $1, 2, \dots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if π is an even permutation and $\sigma(\pi) = -1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π . Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1} .$$

2003-A-5. A Dyck n -path is a lattice path of n upsteps $(1, 1)$ and n downsteps $(1, -1)$ that starts at the origin O and never dips below the x -axis. A return is a maximal sequence of contiguous downsteps that terminates on the x -axis. For example, the Dyck 5-path (up, up, down, up, up, down, down, down, up, down) has two returns of lengths 3 and 1 respectively. Show that there is a one-to-one correspondence between the Dyck n -paths with no return of even length and the Dyck $(n-1)$ -paths.

2002-A-4. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and Player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

2002-B-2. Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

Each player, in turn, signs his name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

2002-B-4. An integer n , unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

2000-B-5. Let S_0 be a finite set of positive integers. We define finite sets S_1, S_2, \dots , of positive integers as follows:

Integer a is in S_{n+1} if and only if exactly one of $a - 1$ or a is in S_n .

Show that there exists infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

1997-A-2. Players $1, 2, 3, \dots, n$ are seated around a table and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who then passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

1996-A-3. Suppose that each of twenty students has made a choice of anywhere from zero to six courses from a total of six courses offered. Prove or disprove: There are five students and two courses such that all five have chosen both courses or all five have chosen neither.

1996-B-1. Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with a proof, the number of subsets of $\{1, 2, \dots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

1996-B-5. Given a finite string S of symbol X and O , we write $\Delta(S)$ for the number of X 's in S minus the number of O 's. For example, $\Delta(XOOXOOX) = -1$. We call a string S **balanced** if every substring T of (consecutive symbols of) S has $-2 \leq \Delta(T) \leq 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n .

1995-A-4. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n - 1$. Prove that we can cut the necklace to form a string whose consecutive labels x_1, x_2, \dots, x_n satisfy

$$\sum_{i=1}^k x_i \leq k - 1 \quad \text{for } k = 1, 2, \dots, n.$$

1995-B-1. For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x . Prove that for any two partitions π and π' , there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S .]

1995-B-5. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking **either**

- a. one bean from a heap, provided at least two beans are left behind in that heap, **or**
- b. a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

1994-A-3. Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

1994-A-6. Let f_1, f_2, \dots, f_{10} be bijections of the set of integers such that for each integer n , there is some composition $f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n . Consider the set of 1024 functions

$$\mathfrak{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \dots \circ f_{10}^{e_{10}} : e_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq 10\}$$

(f_i^0 is the identity function and $f_i^1 = f_i$). Show that if A is any nonempty finite set of integers, then at most 512 of the functions in \mathfrak{F} map A to itself.

1993-A-3. Let \mathfrak{P}_n be the set of subsets of $\{1, 2, \dots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathfrak{P}_n \rightarrow \{1, 2, \dots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^m j^n .$$

1992-B-1. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S . For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S ?