OLYMON PROBLEM OF THE WEEK

Produced by the **Canadian Mathematical Society** and the **Department of Mathematics of the University of Toronto**.

Please send your solutions to

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individually as you solve the problems. Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

For those of you who are looking for practice problems, you can access old Olymon problems and solutions on the website sl www.math.utoronto.ca/barbeau/home.html or *www.cms.math.ca*; on the CMS website, you can also access International Mathematical Talent Search Problems as well as problems posed on the Canadian Open Mathematics Challenge and the Canadian Mathematical Olympiad.

Open Problems of the Week

640. [August 16-22] Suppose that $n \geq 2$ and that, for $1 \leq i \leq n$, we have that $x_i \geq -2$ and all the x_i are nonzero with the same sign. Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) > 1 + x_1 + x_2 + \cdots + x_n \quad ,$$

641. [August 23-30] Observe that $x^2 + 5x + 6 = (x + 2)(x + 3)$ while $x^2 + 5x - 6 = (x + 6)(x - 1)$. Determine infinitely many coprime pairs (m, n) of positive integers for which both $x^2 + mx + n$ and $x^2 + mx - n$ can be factored as a product of linear polynomials with integer coefficients.

642. [August 31-September 5] In a convex polyhedron, each vertex is the endpoint of exactly three edges and each face is a concyclic polygon. Prove that the polyhedron can be inscribed in a sphere.

643. [September 6-12] Let n^2 distinct integers be arranged in an $n \times n$ square array ($n \geq 2$). Show that it is possible to select n numbers, one from each row and column, such that if the number selected from any row is greater than another number in this row, then this latter number is less than the number selected from its column.

644. [September 13-19] Given a point P, a line $\mathfrak{L}$ and a circle $\mathfrak{C}$, construct with straightedge and compasses an equilateral triangle PQR with one vertex at P, another vertex Q on $\mathfrak{L}$ and the third vertex R on $\mathfrak{C}$.

645. [September 20-26] Let $n \geq 3$ be a positive integer. Are there n positive integers $a_1, a_2, \cdots, a_n$ not all the same such that for each i with $3 \leq i \leq n$ we have

$$a_i + S_i = (a_i, S_i) + [a_i, S_i] \quad .$$

where $S_i = a_1 + a_2 + \cdots + a_n$, and where $(\cdot, \cdot)$ and $[\cdot, \cdot]$ represent the greatest common divisor and least common multiple respectively?

646. Let ABC be a triangle with incentre I. Let AI meet BC at L, and let X be the contact point of the incircle with the line BC. If D is the reflection of L on X, we construct B' and C' as the reflections of D with respect to the lines BI and CI, respectively. Show that the quadrailateral BCC'B' is cyclic.

A couple of clarifications: The point L lies on BC and X is the midpoint of DL. The problem comes with a diagram that shows AC is at least as long as AB, so you might want to make this assumption wolog.

647. [September 27-October 3] Find all continuous functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$f(x + f(y)) = f(x) + y$$

for every $x, y \in \mathbf{R}$.

648. [October 4-10] Prove that for every positive integer n, the integer $1 + 5^n + 5^{2n} + 5^{3n} + 5^{4n}$ is composite.

649. [October 11-17] In the triangle ABC, $\angle BAC = 20°$ and $\angle ACB = 30°$. The point M is located in the interior of triangle ABC so that $\angle MAC = \angle MCA = 10°$. Determine $\angle BMC$.

650. [October 18-24] Suppose that the nonzero real numbers satisfy

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{xyz} .$$

Determine the minimum value of

$$\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2} .$$

651. [October 25-31] Determine polynomials $a(t)$, $b(t)$, $c(t)$ with integer coefficients such that the equation $y^2 + 2y = x^3 - x^2 - x$ is satisfied by $(x, y) = (a(t)/c(t), b(t)/c(t))$.

652. [November 1-7] (a) Let m be any positive integer greater than 2, such that $x^2 \equiv 1 \pmod{m}$ whenever the greatest common divisor of x and m is equal to 1. An example is $m = 12$. Suppose that n is a positive integer for which $n + 1$ is a multiple of m. Prove that the sum of all of the divisors of n is divisible by m.

(b) Does the result in (a) hold when $m = 2$?

(c) Find all possible values of m that satisfy the condition in (a).

653. Let $f(1) = 1$ and $f(2) = 3$. Suppose that, for $n \geq 3$, $f(n) = \max\{f(r) + f(n - r) : 1 \leq r \leq n - 1\}$. Determine necessary and sufficient conditions on the pair (a, b) that $f(a + b) = f(a) + f(b)$.

654. Let ABC be an arbitrary triangle with the points D, E, F on the sides BC, CA, AB respectively, so that

$$\frac{BD}{DC} \leq \frac{BF}{FA} \leq 1$$

and

$$\frac{AE}{EC} \leq \frac{AF}{FB} .$$

Prove that $[DEF] \leq \frac{1}{4}[ABC]$, with equality if and only if two at least of the three points D, E, F are midpoints of the corresponding sides.
(*Note:* $[XYZ]$ denotes the area of triangle XYZ.)

655. (a) Three ants crawl along the sides of a fixed triangle in such a way that the centroid (intersection of the medians) of the triangle they form at any moment remains constant. Show that this centroid coincides with the centroid of the fixed triangle if one of the ants travels along the entire perimeter of the triangle.

(b) Is it indeed always possible for a given fixed triangle with one ant at any point on the perimeter of the triangle to place the remaining two ants somewhere on the perimeter so that the centroid of their triangle coincides with the centroid of the fixed triangle?

Problem 646 appears in the latest issue of the College Mathematics Journal: (40:4 September, 2009). If you solve it, you can send solutions both to me and to the problems editor: Professor Shing So, either by email as a pdf TeX or Word attachment to so@ucmo.edu (preferred) or by mail to Shing S. So CMJ Solutions Department of Mathematics and Computer Science University of Central Missouri Warrensburg, MO 64093

The problem is numbered 909 in the Journal and you should refer to it as such in any message that you send to Professor So.

Problems 649 and 650 are set by Rosu Mihai, so you can send your solutions to him by mail at Rosu Mihai, 54 Judith Crescent, Brampton, ON L6S 3J4 or by email at *rosumihai@yahoo.ca*, as well as to *barbeau@math.utoronto.ca*.