OLYMON

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PROBLEMS FOR MAY

Please send your solution to
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no later than June 15, 2005 It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

381. Determine all polynomials $f(x)$ such that, for some positive integer k,

$$f(x^k) - x^3 f(x) = 2(x^3 - 1)$$

for all values of x.

382. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

383. Place the numbers $1, 2, \cdots, 9$ in a 3×3 unit square so that
(a) the sums of numbers in each of the first two rows are equal;
(b) the sum of the numbers in the third row is as large as possible;
(c) the column sums are equal;
(d) the numbers in the last row are in descending order.

Prove that the solution is unique.

384. Prove that, for each positive integer n,

$$(3 - 2\sqrt{2})(17 + 12\sqrt{2})^n + (3 + 2\sqrt{2})(17 - 12\sqrt{2})^n - 2$$

is the square of an integer.

385. Determine the minimum value of the product $(a + 1)(b + 1)(c + 1)(d + 1)$, given that $a, b, c, d \geq 0$ and

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 1 \ .$$

386. In a round-robin tournament with at least three players, each player plays one game against each other player. The tournament is said to be *competitive* if it is impossible to partition the players into two sets, such that each player in one set beat each player in the second set. Prove that, if a tournament is not competitive, it can be made so by reversing the result of a single game.

387. Suppose that a, b, u, v are real numbers for which $av - bu = 1$. Prove that

$$a^2 + u^2 + b^2 + v^2 + au + bv \geq \sqrt{3} \ .$$

Give an example to show that equality is possible. (Part marks will be awarded for a result that is proven with a smaller bound on the right side.)