Solutions to the 2006 CMO paper

1. Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most 2 candies. For example, if $n=3$, then $f(3,7)=0$, $f(3,6)=1$ and $f(3,4)=6$.

   Determine the value of

$$f(2006,1)+f(2006,4)+f(2006,7)+\cdots+f(2006,1000)+f(2006,1003)\ .$$

Comment. Unfortunately, there was an error in the statement of this problem. It was intended that the sum should continue to $f(2006,4012)$.

Solution 1. The number of ways of distributing k candies to 2006 children is equal to the number of ways of distributing 0 to a particular child and k to the rest, plus the number of ways of distributing 1 to the particular child and $k-1$ to the rest, plus the number of ways of distributing 2 to the particular child and $k-2$ to the rest. Thus $f(2006,k)=f(2005,k)+f(2005,k-1)+f(2005,k-2)$, so that the required sum is

$$1+\sum_{k=1}^{1003} f(2005,k)\ .$$

In evaluating $f(n,k)$, suppose that there are r children who receive 2 candies; these r children can be chosen in $\binom{n}{r}$ ways. Then there are $k-2r$ candies from which at most one is given to each of $n-r$ children. Hence

$$f(n,k)=\sum_{r=0}^{\lfloor k/2\rfloor}\binom{n}{r}\binom{n-r}{k-2r}=\sum_{r=0}^{\infty}\binom{n}{r}\binom{n-r}{k-2r}\ ,$$

with $\binom{x}{y}=0$ when $x<y$ and when $y<0$. The answer is

$$\sum_{k=0}^{1003}\sum_{r=0}^{\infty}\binom{2005}{r}\binom{2005-r}{k-2r}=\sum_{r=0}^{\infty}\binom{2005}{r}\sum_{k=0}^{1003}\binom{2005-r}{k-2r}\ .$$

Solution 2. The desired number is the sum of the coefficients of the terms of degree not exceeding 1003 in the expansion of $(1+x+x^2)^{2005}$, which is equal to the coefficient of x^{1003} in the expansion of

$$\begin{aligned}(1+x+x^2)^{2005}(1+x+\cdots+x^{1003})&=[(1-x^3)^{2005}(1-x)^{-2005}](1-x^{1004})(1-x)^{-1}\\&=(1-x^3)^{2005}(1-x)^{-2006}-(1-x^3)^{2005}(1-x)^{-2006}x^{1004}\ .\end{aligned}$$

Since the degree of every term in the expansion of the second member on the right exceeds 1003, we are looking for the coefficient of x^{1003} in the expansion of the first member:

$$\begin{aligned}(1-x^3)^{2005}(1-x)^{-2006}&=\sum_{i=0}^{2005}(-1)^i\binom{2005}{i}x^{3i}\sum_{j=0}^{\infty}(-1)^j\binom{-2006}{j}x^j\\&=\sum_{i=0}^{2005}\sum_{j=0}^{\infty}(-1)^i\binom{2005}{i}\binom{2005+j}{j}x^{3i+j}\\&=\sum_{k=0}^{\infty}\left(\sum_{i=1}^{2005}(-1)^i\binom{2005}{i}\binom{2005+k-3i}{2005}\right)x^k\ .\end{aligned}$$

The desired number is

$$\sum_{i=1}^{334}(-1)^i\binom{2005}{i}\binom{3008-3i}{2005}=\sum_{i=1}^{334}(-1)^i\frac{(3008-3i)!}{i!(2005-i)!(1003-3i)!}\ .$$

(Note that $\binom{3308-3i}{2005} = 0$ when $i \geq 335$.)

2. Let ABC be an acute-angled triangle. Inscribe a rectangle $DEFG$ in this triangle so that D is on AB, E is on AC and both F and G are on BC. Describe the locus of (*i.e.*, the curve occupied by) the intersections of the diagonals of all possible rectangles $DEFG$.

Solution. The locus is the line segment joining the midpoint M of BC to the midpoint K of the altitude AH. Note that a segment DE with D on AB and E on AC determines an inscribed rectangle; the midpoint F of DE lies on the median AM, while the midpoint of the perpendicular from F to BC is the centre of the rectangle. This lies on the median MK of the triangle AMH.

Conversely, any point P on MK is the centre of a rectangle with base along BC whose height is double the distance from K to BC.

3. In a rectangular array of nonnegative real numbers with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

Solution 1. Consider first the case where all the rows have the same positive sum s; this covers the particular situation in which $m = 1$. Then each column, sharing a positive element with some row, must also have the sum s. Then the sum of all the entries in the matrix is $ms = ns$, whence $m = n$.

We prove the general case by induction on m. The case $m = 1$ is already covered. Suppose that we have an $m \times n$ array not all of whose rows have the same sum. Let $r < m$ of the rows have the sum s, and each of the of the other rows have a different sum. Then every column sharing a positive entry with one of these rows must also have sum s, and these are the only columns with the sum s. Suppose there are c columns with sum s. The situation is essentially unchanged if we permute the rows and then the column so that the first r rows have the sum s and the first c columns have the sum s. Since all the entries of the first r rows not in the first c columns and in the first c columns not in the first r rows must be 0, we can partition the array into a $r \times c$ array in which all rows and columns have sum s and which satisfies the hypothesis of the problem, two rectangular arrays of zeros in the upper right and lower left and a rectangular $(m-r) \times (n-c)$ array in the lower right that satisfies the conditions of the problem. By the induction hypothesis, we see that $r = c$ and so $m = n$.

Solution 2. [Y. Zhao] Let the term in the ith row and the jth column of the array be denoted by a_{ij}, and let $S = \{(i,j) : a_{ij} > 0\}$. Suppose that r_i is the sum of the ith row and c_j the sum of the jth column. Then $r_i = c_j$ whenever $(i,j) \in S$. Then we have that

$$\sum\{\frac{a_{ij}}{r_i} : (i,j) \in S\} = \sum\{\frac{a_{ij}}{c_j} : (i,j) \in S\} \ .$$

We evaluate the sums on either side independently.

$$\begin{aligned}\sum\{\frac{a_{ij}}{r_i} : (i,j) \in S\} &= \sum\{\frac{a_{ij}}{r_i} : 1 \leq i \leq m, 1 \leq j \leq n\} \\ &= \sum_{i=1}^{m} \frac{1}{r_i} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m} \left(\frac{1}{r_i}\right) r_i \\ &= \sum_{i=1}^{m} 1 = m \ .\end{aligned}$$

$$\begin{aligned}\sum\{\frac{a_{ij}}{c_j} : (i,j) \in S\} &= \sum\{\frac{a_{ij}}{c_j} : 1 \leq i \leq m, 1 \leq j \leq n\} \\ &= \sum_{j=1}^{n} \frac{1}{c_j} \sum_{i=1}^{m} a_{ij} = \sum_{j=1}^{n} \left(\frac{1}{c_j}\right) c_j \\ &= \sum_{j=1}^{n} 1 = n \ .\end{aligned}$$

Hence $m = n$.

Comment. The second solution can be made cleaner and more elegant by defining $u_{ij} = a_{ij}/r_i$ for all (i, j). When $a_{ij} = 0$, then $u_{ij} = 0$. When $a_{ij} > 0$, then, by hypothesis, $u_{ij} = a_{ij}/c_j$, a relation that in fact holds for all (i, j). We find that

$$\sum_{j=1}^{n} u_{ij} = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} u_{ij} = 1$$

for $1 \le i \le m$ and $1 \le j \le n$, so that (u_{ij}) is an $m \times n$ array whose row sums and column sums are all equal to 1. Hence

$$m = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} u_{ij} \right) = \sum \{u_{ij} : 1 \le i \le m, 1 \le j \le n\} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} u_{ij} \right) = n$$

(being the sum of all the entries in the array).

4. Consider a round-robin tournament with $2n + 1$ teams, where each team plays each other team exactly once. We say that three teams X, Y and Z, form a *cycle triplet* if X beats Y, Y beats Z, and Z beats X. There are no ties.

   (a) Determine the minimum number of cycle triplets possible.

   (b) Determine the maximum number of cycle triplets possible.

Solution 1. (a) The minimum is 0, which is achieved by a tournament in which team T_i beats T_j if and only if $i > j$.

(b) Any set of three teams constitutes either a cycle triplet or a "dominated triplet" in which one team beats the other two; let there be c of the former and d of the latter. Then $c + d = \binom{2n+1}{3}$. Suppose that team T_i beats x_i other teams; then it is the winning team in exactly $\binom{x_i}{2}$ dominated triples. Observe that $\sum_{i=1}^{2n+1} x_i = \binom{2n+1}{2}$, the total number of games. Hence

$$d = \sum_{i=1}^{2n+1} \binom{x_i}{2} = \frac{1}{2} \sum_{i=1}^{2n+1} x_i^2 - \frac{1}{2} \binom{2n+1}{2} .$$

By the Cauchy-Schwarz Inequality, $(2n + 1) \sum_{i=1}^{2n+1} x_i^2 \ge (\sum_{i=1}^{2n+1} x_i)^2 = n^2(2n + 1)^2$, whence

$$c = \binom{2n+1}{3} - \sum_{i=1}^{2n+1} \binom{x_i}{2} \le \binom{2n+1}{3} - \frac{n^2(2n+1)}{2} + \frac{1}{2}\binom{2n+1}{2} = \frac{n(n+1)(2n+1)}{6} .$$

To realize the upper bound, let the teams be $T_1 = T_{2n+2}$, $T_2 = T_{2n+3}$. $\cdots$, $T_i = T_{2n+1+i}$, $\cdots$, $T_{2n+1} = T_{4n+2}$. For each i, let team T_i beat $T_{i+1}, T_{i+2}, \cdots, T_{i+n}$ and lose to $T_{i+n+1}, \cdots, T_{i+2n}$. We need to check that this is a consistent assignment of wins and losses, since the result for each pair of teams is defined twice. This can be seen by noting that $(2n + 1 + i) - (i + j) = 2n + 1 - j \ge n + 1$ for $1 \le j \le n$. The cycle triplets are $(T_i, T_{i+j}, T_{i+j+k})$ where $1 \le j \le n$ and $(2n + 1 + i) - (i + j + k) \le n$, *i.e.*, when $1 \le j \le n$ and $n + 1 - j \le k \le n$. For each i, this counts $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$ cycle triplets. When we range over all i, each cycle triplet gets counted three times, so the number of cycle triplets is

$$\frac{2n+1}{3}\left(\frac{n(n+1)}{2}\right) = \frac{n(n+1)(2n+1)}{6} .$$

Solution 2. [S. Eastwood] (b) Let t be the number of cycle triplets and u be the number of ordered triplets of teams (X, Y, Z) where X beats Y and Y beats Z. Each cycle triplet generates three ordered triplets while other triplets generate exactly one. The total number of triplets is

$$\binom{2n+1}{3} = \frac{n(4n^2-1)}{3} .$$

The number of triples that are not cycle is

$$\frac{n(4n^2-1)}{3} - t \ .$$

Hence

$$u = 3t + \left(\frac{n(4n^2-1)}{3} - t \right) \Longrightarrow$$

$$t = \frac{3u - n(4n^2-1)}{6} = \frac{u - (2n+1)n^2}{2} + \frac{n(n+1)(2n+1)}{6} \ .$$

If team Y beats a teams and loses to b teams, then the number of ordered triples with Y as the central element is ab. Since $a + b = 2n$, by the Arithmetic-Geometric Means Inequality, we have that $ab \leq n^2$. Hence $u \leq (2n+1)n^2$, so that

$$t \leq \frac{n(n+1)(2n+1)}{6} \ .$$

The maximum is attainable when $u = (2n+1)n^2$, which can occur when we arrange all the teams in a circle with each team beating exactly the n teams in the clockwise direction.

Comment. Interestingly enough, the maximum is $\sum_{i=1}^{n} i^2$; is there a nice argument that gives the answer in this form?

5. The vertices of a right triangle ABC inscribed in a circle divide the circumference into three arcs. The right angle is at A, so that the opposite arc BC is a semicircle while arc AB and arc AC are supplementary. To each of the three arcs, we draw a tangent such that its point of tangency is the midpoint of that portion of the tangent intercepted by the extended lines AB and AC. More precisely, the point D on arc BC is the midpoint of the segment joining the points D' and D'' where the tangent at D intersects the extended lines AB and AC. Similarly for E on arc AC and F on arc AB.

   Prove that triangle DEF is equilateral.

   `Take in diagram`

Solution 1. A prime indicates where a tangent meets AB and a double prime where it meets AC. It is given that $DD' = DD''$, $EE' = EE''$ and $FF' = FF''$. It is required to show that arc EF is a third of the circumference as is arc DBF.

AF is the median to the hypotenuse of right triangle $AF'F''$, so that $FF' = FA$ and therefore

$$\begin{aligned} \text{arc } AF &= 2\angle F''FA = 2(\angle FF'A + \angle FAF') \\ &= 4\angle FAF' = 4\angle FAB = 2 \text{ arc } BF \ , \end{aligned}$$

whence arc $FA = (2/3)$ arc BFA. Similarly, arc $AE = (2/3)$ arc AEC. Therefore, arc FE is 2/3 of the semicircle, or 1/3 of the circumference as desired.

As for arc DBF, arc $BD = 2\angle BAD = \angle BAD + \angle BD'D = \angle ADD'' = (1/2)$ arc ACD. But, arc $BF = (1/2)$ arc AF, so arc $DBF = (1/2)$ arc $FAED$. That is, arc DBF is 1/3 the circumference and the proof is complete.

Solution 2. Since $AE'E''$ is a right triangle, $AE = EE' = EE''$ so that $\angle CAE = \angle CE''E$. Also $AD = D'D = DD''$, so that $\angle CDD'' = \angle CAD = \angle CD''D$. As $EADC$ is a concyclic quadrilateral,

$$\begin{aligned} 180^\circ &= \angle EAD + \angle ECD = \angle DAC + \angle CAE + \angle ECA + \angle ACD \\ &= \angle DAC + \angle CAE + \angle CEE'' + \angle CE''E + \angle CDD'' + \angle CD''D \\ &= \angle DAC + \angle CAE + \angle CAE + \angle CAE + \angle CAD + \angle CAD \\ &= 3(\angle DAC + \angle DAE) = 3(\angle DAE) \ , \end{aligned}$$

Hence $\angle DFE = \angle DAE = 60^\circ$. Similarly, $\angle DEF = 60^\circ$. It follows that triangle DEF is equilateral.