**Mathematics in a deck of cards**

While the acquisition of skills is important, pupils of mathematics also need an educational regime that authentically conveys to them other aspects of mathematics, in particular the way in which observations can be organized and analyzed. Students should be presented with situations in which structure is visible and can be studied. As mathematicians, we have faith that patterns and phenomena can be understood, and a decent curriculum should provide occasions for demonstrating this.

One vehicle with the young is an ordinary deck of 52 playing cards, with its thirteen ranks and four suits. I will suggest some interactions between a "magician" and his "subject".

**1. Three questions for 27 options.** The magician deals 27 cards into three 9-card columns and asks the subject to secretly select one of the cards, but tell him which column contains it. Once the magician has this information, he gathers up to three columns, one on top of the next, and then deals the cards across into three 9-card columns. He then ascertains from the subject which column contains the selected card and again deals the cards across into three 9-card columns. Upon being told a third time which column contains the selected card, he is able to identify it.

The trick is based on dealing out the cards so that the first answer narrows the selected card down to one of nine cards, the second answer to one of three cards and the third answer down to a unique card. This trick is known to many youngsters, sometimes in the form of dealing only 7 cards to a column. Often it is set up, so that the named column is gathered up in the middle, so that the selected card turns out to be in the very middle of the deck.

It is possible for many children from the age of 9 to understand and replicate the trick, after the magician walks them through it a couple of times. All that is required is a sufficient level of concentration to keep track of where the nine cards of each selected column go and to make sure that three of them are dealt into each of the three columns the next time around.

The surprise comes from fact that one can isolate one of 27 possibilities with three questions; the cube of 3 is as *big* as 27. The same perspective applied to base ten numeration; it takes only four pieces of information to specify a number less than 10000, namely its four digits. This trick, thus, can possibly alter perceptions, something desirable in a mathematics class.

**2. The flipover.** Select the ten hearts from ace to ten, inclusive, and arrange them in increasing order in a fan. The magician presents the fan, cards face down, to the subject and asks her to pull out two *adjacent* cards, turn them over and reinsert them face up into the spot whence they were taken. Thus, if 4♡ and 5♡ were removed, the 5 will be where the 4 was, face up, and vice versa. He asks the subject to continue performing several times the following: cut the deck and put one end before the other, and pull out two adjacent cards, turn them over and restore them in place (either card chosen can be face down or face up).

Then the magician does something sight unseen by either person and then shows the fan; all the even cards are facing one way and the odd cards the other. What has the magician done, and why does it work?

The key to this is that parity of the cards in the fan alternate, and the actions, in a more general sense, preserve the alternation. Since cutting the deck is like moving the cards around in a ring, we will assume the cards start face down in a ring, ignore the cut, and just focus on the turnover. In each position in the ring, the cards are in one of two states $EU - OD$ (even-up, odd-down) or $ED - OU$ (even-down, odd-up). These states alternate with position, and continue to alternate with each flip. Turning over a single card and restoring it into the same position reverses the state of that card.

To give a hint to the children, one might point out that whatever the magician did at the end should work if no operations at all were carried out.

**3. Still complete in the halves.** Two packs of 13 cards, one consisting of the 13 spades in order from ace to king and the other consisting of the 13 hearts in reverse order from king to ace are placed face down on the table and subjected to a rough riffle shuffle. This means that they are incorporated into a single pile, with cards incorporated in bunches alternately from the two packs. (For a perfect riffle, the cards are mixed one alternately from each pack.)

The top thirteen cards are taken from the united pack. It turns out that each of the ranks from ace to king appears exactly once among them. The same is true for the pile left behind. Why is that?

Note that in the incorporated pack, the hearts and spades remain in the same order; they are just interspersed. Suppose, for example, that the top thirteen cards contain no six of spades. Then at most five spades made it into the top thirteen, the ace through five. So at least eight hearts must be there, the king through six. Thus, the six of hearts must be present.

**4. Picking the correct pair.** The magician deals onto the table ten pairs of cards, and asks the subject to select one of the pairs silently. The magician then gathers the pairs up and deals them into four rows of five cards each. Upon being told which rows contain the two cards of the chosen pair, the magician can identify them.

This is easy to explain, as it simply depends on producing a one-one correspondence between the ten pairs and the number of ways of picking two rows out of four, with the possibility of a row being selected twice. The magician picks the cards up keeping the pairs together, and then carefully deals each pair into two particular rows. For example, the ten pairs can be dealt into rows (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4).

A less transparent way of dealing into rows is possible. Keep in mind the four words ATLAS, BIBLE, GOOSE and THIGH. The words have ten different letters, each occurring exactly twice. Each letter appears in a different pair of the words, and each pair of words has exactly one letter in common (with each word having one letter appearing twice). Cued by these words, you can deal the pairs accordingly.

**5. Go to the top!** A pack of the thirteen spades is thoroughly shuffled and the cards are laid out from left to right on the table. We adopt the usual convention that $A = 1$, $J = 11$, $Q = 12$ and $K = 13$. If the leftmost card is $k$, then the $k$th card from the left is taken from its position and placed in the first position at the left. The order of the remaining cards is left undisturbed. This move is repeated. It is found that, regardless of the original order of the cards, eventually the ace is brought to the left and the process stops. Why is this?

This probably needs to be performed a few times until the students begin to see the dynamic. Basically, the ace either stays in its original position, or gets shoved to the right, until it is suddenly brought to the leftmost position. If the ace starts out in the $n$th position, then one of the left $n - 1$ cards must have rank $n$ or bigger. One needs to argue that one such card eventually gets “hit”, whereupon the ace either comes to the first position or moves one position to the right. This is a nice example for discussion of induction.

**6. Which card comes last?** The magician takes 16 cards from the deck and places them upsidedown in a stack on the table. The subject is asked to remove from the top fewer than half of them, leaving a stack of between 9 and 15 cards. The magician then picks this up and shows the subject (but not himself) the cards in the stack. If the subject removed $k$ cards, the subject is asked to remember the value of the $k$th card from the bottom.

The magician then takes up the stack, cards upsidedown, and deals the cards alternately to the bottom of the stack and face-up onto the table until only one card remains in the stack. This card turns out to be the one identified earlier by the subject.

This is a manifestation of a Josephus situation; a group of people are arranged in a circle, and each $r$th person is eliminated until only one remains. Here $r = 2$. In the present situation, suppose that $n$ individuals numbered from 1 to $n$ are in a ring, and we start with individual 1 and eliminate every second one as we count around. If $f(n)$ is the last individual to remain, we can see that $f(2^m) = 1$ for every nonnegative integer $m$ and that $f(n + 1) = f(n) + 2$ when $n + 1$ is not a power of 2. Then $f(16 - k) = 17 - 2k = (16 - k) - (k - 1)$ (for $1 \leq k \leq 7$), so that the final card is the $k$th card from the end of the $16 - k$ cards.

**8. A little hidden algebra.** The magician takes 26 cards from a regular deck and places it face down on the table. He then turns over the cards one by one to show the subject that the deck is randomly mixed, and then restores the 26 cards to the original position; call this the *stock*. Handing the remaining 26 cards to the subject, he instructs the subject to place a card face up on the table. We will use the equivalence $A = 1$, $J = Q = K = 10$. If the card turned up is $k$, the subject then places on top of it sufficiently many cards face up to count up to ten. The ranks of the additional cards are immaterial, the subject counting $k$, $k + 1$, $\cdots$, 10 until she reaches 10. Then the subject starts a new pile by placing one of the remaining cards on the table, and performing the same operation. This is repeated as long as there are sufficiently many cards and there are at least three piles. (In the rare case that there are not enough cards to form three piles, the subject can "borrow" from the top of the stock.)

The subject then turns three of the piles over and puts the rest of the cards face down on top of the 26-card stock left by the magician. The subject is then to turn over the top card on each of the three piles, add them and count down that many cards in the stock (the 26-cards augmented by the leftovers). While the subject is doing this, the magician predicts what the terminal card will be.

For example, suppose the subject turns over a 4; then she will place on top of it face up six more cards, counting as she goes 5 - 6 - 7 - 8 - 9 - 10. If the three piles chosen are built on, say, 4, 3 and 8, then the three piles built up on them will have, respectively, 7, 8 and 3 cards. Eight cards will be returned to the stock, which will now have a total of 34 cards. When the subject turns over the three piles and reveals the top cards, these will, of course, be 4, 3 and 8, and the subject will count down 15 cards into the stock. This will go through the eight returned to the stock and end up with the seventh from the top of the original stock of 26.

Remarkably, no matter what cards are turned face up, the count will go down to the seventh card from the top of the 26-card stock, and it is this card that the magician must memorize. I usually convince students that it works in the following way. Suppose that the three cards turned up are all tens. Then twenty three cards are returned to the stock, and we have to count down 30 cards to the seventh from the top of the original stock. For every reduction of one in the sum of the three cards, there is one more card in the three piles and one fewer returned to the stock. At the same time, there is one fewer card to count down, so we will always wind up in the same place.

I am indebted to Peter Taylor of Queen's University for showing me this nice trick.

**9. A quick reversion to order.** Begin with a new deck of cards in which the suits appear in order, ranked in order. A remarkable fact is that eight perfect inside riffle shuffles (where the top and bottom cards of the deck remain in position) will restore the deck to its original order. If, like me, you cannot perform a perfect riffle shuffle, you can deal them to obtain the inverse effect of a riffle and still get a striking effect. Suppose that the cards are numbered from 0 to 51, inclusive, and are originally in this order from top to bottom. Deal the cards face up alternately into left and right piles, 0 to the left, 1 to the right, and so on. Pick up the piles, putting the right pile on the left one, turn the incorporated deck upside down and repeat. Now 0 goes to the left, 2 to the right, 4 to the left, 6 to the right and so on. Repeat the process.

Each time the process is repeated and the deck incorporated, the value of the card in any given position gets multiplied by 2 modulo 51. Since $2^8 \equiv 1 \pmod{51}$, eight repetitions will bring the cards back to the original order. However, when the cards are dealt face up, students can see how the order changes from one deal to the next and some interesting things occur. Try it!

While one generally cannot go into the number theory involved for most school students, the investigation of how long it takes this shuffle to return a deck to its original order for various numbers of cards is worthwhile.

**Pedagogical considerations.** Do such card stunts have a place in the curriculum? Most assuredly they do. Apart from the "fun" aspect, there is real mathematics here. None of these involve sleight of hand or any motor skills; they can be carried out by any student. They are all mathematically based, and can be justified through a careful analysis that is accessible, in some cases, even to elementary students. Their value in the curriculum is that they give an authentic view of the analytical side of mathematics that the standard syllabus, with its emphasis on skills, either hardly hints at or obscures with technicalities. In analyzing the arguments for the tricks, one can see that important mathematical ideas, such as pairing, induction, algebraic structure and transformations are adumbrated.

Even though formal proofs might not be appropriate, enough can be said to convince students of what makes the tricks work. The important message is that of the possibility of proof and the adoption on a perspective that helps to see what is going on. The more technical aspects of the construction and presentation of proofs will not come later on in a vacuum.

Because of the difficulty of systematizing and testing such activities, it may be though that they are not suitable in a curriculum. But this is a strong argument for inclusion. Any attempt to formalize or test them would be destructive. It can be argued that some of the most important things we want to convey about the mathematical enterprise are things that cannot and ought not to be tested, but rather insinuated where appropriate into the regular mathematical experiences of the students, so that they become part of the landscape.

Like all attempts to alter the thrust of the curriculum, this will succeed or fail depending on the background and quality of he teaching corps. This is another instance of how we must start with sound policies for the recruitment and formation of teachers before we can contemplate the reforms in mathematical schooling we would all like to see.

*Ed Barbeau*