Random Walks and Geometry on Graphs of Exponential Growth

by

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Abstract

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We study the relationship between random walk and geometry on graphs of exponential growth. In the first chapter we introduce a tool that might be useful in other settings: a new lower bound for the hitting time of a set for a random walk on a graph in terms of distance, volume and electric resistance.

The second chapter uses this result to give upper bounds for the linear rate of escape of the random walk in terms of the geometry of the graph, answering questions posed by Lyons, Pemantle and Peres (1997). For trees, we get a bound for the speed in terms of the Hausdorff dimension of the harmonic measure on the boundary. As a consequence, two conjectures of Lyons, Pemantle and Peres are resolved, and a new bound is given for the dimension of the harmonic measure defined by the biased random walk on a Galton-Watson tree.

The third chapter studies anchored expansion, a non-uniform version of the strong isoperimetric inequality. We show that every graph with i-anchored expansion contains a subgraph with isoperimetric (Cheeger) constant of at least i. We prove a conjecture by Benjamini, Lyons and Schramm (1999) that in such graphs the random walk escapes with a positive lim inf speed. We also show that anchored expansion

implies that the heat-kernel decays at a sub-exponential rate of at least $\exp(-cn^{1/3})$.

Professor Yuval Peres
Dissertation Committee Chair

Introduction

One of the most surprising facts for a newcomer to probability theory is that random walk on the Euclidean lattice is recurrent in two dimensions and transient in three. Of similar interest is the observation that on a lattice in the hyperbolic plane the walk is not only transient but escapes at a positive limiting speed. The same happens on a regular tree, and in fact on any graph that grows "fast enough" in a uniform sense. The goal of this work is to study such rapidly growing graphs and to relate geometric measures of growth to properties of the random walk.

The first chapter introduces a new lower bound for hitting times of sets in finite graphs. This bound applies both to expected value and to lower large deviations, and is extensively used in the second chapter.

The second chapter addresses the following riddle: can one rearrange the branching structure of a binary tree so that in the new tree the limiting speed of the random walk will be with positive probability faster than in the original tree? We give upper bounds on the limiting speed of the random walk on a graph in terms of different measures of growth. Some applications to Galton-Watson trees are also discussed.

The third chapter studies a robust version of the strong isoperimetric inequality, anchored expansion. We show that this property implies that the walk escapes at a positive speed and that the heat-kernel decays at least sub-exponentially of order $\exp(-cn^{1/3})$.

Bibliographical note. The first chapter is taken from the preprint Virág (2000c). The second chapter contains results from Virág (2000b), significantly improved and shortened thanks to the new results in the first chapter. The third chapter contains results from Virág (2000a) with minor modifications to unify notation.

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Chapter 1

A lower bound for hitting times

1.1 Introduction

The goal of this chapter is to give a lower bound for the hitting times of sets for reversible Markov chains. Every reversible Markov chain can be represented as random walk on weighted graph $G = (V, w)$ that is a set of vertices with a symmetric nonnegative edge function w on $V \times V$. At each step, the walk chooses a neighboring site at random with odds given by the edge weights, and then moves there. The weight w_x of a vertex x can be defined as the sum of the weights over all incident edges. Define the weight w_z of a vertex set z as the sum of the vertices in the set.

The graph G can be though of as an electric network, with conductances given by the edge weights. Let r_{oz} denote the effective resistance between vertices or vertex sets o, z. Let T_{oz} denote the hitting time of z for the walk started at the vertex o. The following classical result (Chandra et al. 1989) is often used for bounding expected hitting times:

$$\mathbf{E}T_{oz} + \mathbf{E}T_{zo} = w_V r_{oz}.$$

Our main result is that a related quantity, $w_z r_{oz}$ can be used to bound the expected value and lower large deviation probabilities of the hitting time T_{oz}. These bounds also work, and are most useful, when z denotes a vertex set of large volume.

Theorem 1 *Let o be a vertex and z be a vertex set in a weighted graph satisfying*

$\mathrm{dist}(o,z) \geq n+1$ *for some integer* $n \geq 0$. *Then*

$$\begin{aligned} \mathbf{E}T_{oz} &\geq mn+1, \\ \mathbf{P}(T_{oz} \leq an+1) &\leq e^{-I(a)n} \end{aligned}$$

for $a \in [1,m]$, *where* m, I *denote the mean and the large deviation rate function of the hitting time* T'_{01} *for biased simple random walk on the integers with odds* 1 *and* $g := (w_z r_{oz})^{1/n}$ *of going left and right, respectively. The explicit formulae are*

$$m = \mathbf{E}T'_{01} = (g+1)/(g-1), \tag{1}$$

$$e^{-I(a)} = (a-1)\left(\frac{a^2-1}{g}\right)^{\frac{a-1}{2}}\left(\frac{g+1}{2a}\right)^a. \tag{2}$$

For easy reference, we give two simple bounds for the rate function in (19). The above theorem will follow as a corollary to

Proposition 2 *The Laplace transform of* $T'_{0n}+1$ *dominates the Laplace transform of* T_{oz}, *that is, for* $\lambda \geq 0$ *we have* $\mathbf{E}e^{-\lambda(T'_{0n}+1)} \geq \mathbf{E}e^{-\lambda T_{oz}}$.

Note that the quantity $w_z r_{oz}$ is the product of w_z/w_o and $w_o r_{oz}$, and the first factor is often dominant. For example, in a graph with unit edge weights, $r_{oz} \leq \mathrm{dist}(o,z)$, which can be much smaller than w_z. Another example can be found in Theorems 6 and 7 in the next Chapter.

1.2 Outline of the proof

The vertices in z may be identified as a single vertex (still denoted z), as it will not change any of the quantities compared. Also, for each $\varepsilon > 0$ there is a finite subgraph G' of G such that up to time ε^{-1} the walk on G and the finite subgraph are the same with probability at least $1-\varepsilon$, and the parameter $w_z r_{oz}$ in the two graphs G and G' differ by at most ε. This means that it suffices to prove the theorem for finite weighted graphs.

A **stopped random walk law** is a quadruple (K,μ,V,z), where V is a finite set of vertices, K is a reversible transition kernel on V, μ is an initial distribution, and $z \in V$ is a special vertex at which the random walk is stopped.

If (K, μ, V, z) is a stopped random walk law, denote K_z the transition kernel of the walk killed at z, that is $K_z(x, y) := K(x, y)\mathbf{1}(x \neq z)$. In addition, for each $\beta \in (0, 1]$ we introduce a version of the killed walk. This walk moves as the walk defined by K_z, but is killed before each step with probability $1 - \beta$; its kernel is denoted $K_\beta(x, y) := \beta K_z(x, y)$.

Let $\mathcal{G}_\beta(x, y)$ denote the Green kernel defined by K_β, i.e. the expected number of times the walk started at x visits y: $\mathcal{G}_\beta(x, y) := \sum_{n=0}^{\infty} K_\beta^n(x, y)$. For a stopped RW law for which the initial distribution is concentrated on a single vertex o, define the following parameters:

$$\Gamma_\beta := \mathcal{G}_\beta(o, o) w_z^o / w_o, \tag{3}$$

$$S_\beta := \mathcal{G}_\beta(o, z). \tag{4}$$

Here w_z^o denotes the total weight on the set of edges incident to z and accessible from o without passing through z. Each initial measure μ is a convex combination of point masses, and we extend the definitions (3, 4) linearly to include all initial distributions.

The parameter S_β gives the probability that the walk survives to hit z, and so is closely related to $T_{\mu z}$ (the hitting time of z starting from distribution μ):

$$S_\beta = \sum_{k=0}^{\infty} \mathbf{P}(T_{\mu z} = k)\beta^k = \mathbf{E}\beta^{T_{\mu z}}.$$

This means that S_β, as a function of $-\log \beta$, is the Laplace transform of $T_{\mu z}$.

From the connection between random walks and electric networks we know that the probability that a random walk on a graph started at vertex o visits vertex z before returning to o is given by w_o/r_{oz}. The number of hits to o before hitting z is therefore a geometric random variable with success probability w_o/r_{oz}, so its expected value is $\mathcal{G}_1(o, o) = r_{oz}/w_o$. Therefore the parameter Γ_β satisfies

$$\Gamma_\beta \leq \Gamma_1 = w_z^o r_{oz} \leq w_z r_{oz}. \tag{5}$$

In the coming sections, we will give an upper bound on S_β in terms of Γ_β for each β. Our main tool is the following proposition, which shows that complicated chains can be simplified without changing the relevant parameters. If (K_i, μ_i, V_i, z_i), $i \in \mathbb{N}$ are stopped RW laws for which the vertex sets $V_i \backslash \{z_i\}$ are disjoint and α_i are nonnegative constants which sum to 1, then a **convex combination** can be defined by picking i at random according to the probabilities α_i, and then running the chain defined by (K_i, μ_i, V_i, z_i) (the vertices z_i can be identified as a single vertex). Since the $V_i \backslash \{z_i\}$ are disjoint, such a combination will still be Markov. It is easy to see that the parameters S_β and Γ_β behave linearly with respect to such convex combinations.

Define a **RW law** as a stopped random walk law for which the probability of getting to z is 0. For such walks, the parameters S_β and Γ_β are identically 0. A stopped RW law is **supported on a path** if the corresponding graph structure is a finite path, μ is unit mass at one endpoint, and z equals the other endpoint.

Proposition 3 (Decomposition of stopped RWs)
Let (K, μ, V, z) be a stopped RW law and let $\beta \in (0,1)$. There exists a stopped RW law (K', μ', V', z') which is a convex combination of stopped RW laws supported on paths and a RW law so that

- *the parameters Γ_β, S_β of (K, μ, V, z), (K', μ', V', z') agree and*
- $\mathrm{dist}_K(\mathrm{supp}(\mu), z) \leq \mathrm{dist}_{K'}(\mathrm{supp}(\mu'), z')$.

It will be relatively easy to compare the parameters S_β and Γ_β for the decomposed chains (Section 1.5). Proposition 2, and so Theorem 1 will then follow easily from Proposition 3, which is proved in Section 1.4. The proof depends on a duality between stopped random walk laws and certain lossy flows presented in the next section.

1.3 Random walks and flows

Let (K, μ, V, z) be a stopped random walk law, and let $\beta \in (0,1)$. Define the function $f : V \times V \to \mathbb{R}_{\geq 0}$

$$f(x,y) := \sum_{v \in V} \mathcal{G}_\beta(v,x) K_\beta(x,y) \mu(v), \tag{6}$$

in words, the expected number of steps the walk defined by K_β makes from x to y. It is easy to see by reversibility of K that for each cycle $\pi = (x_0, ..., x_\ell = x_0)$ and its reversal π' the function f satisfies

$$f(\pi) = f(\pi') \tag{7}$$

Here, and in the sequel, a **function from $V \times V$ applied to a path** will mean the product of the values of the function over the edges of the path.

By comparison of the expected number of steps entering and leaving a vertex $x \in V$ the following **node law** holds:

$$\beta(f(V,x) + \mu(x))\mathbf{1}(x \neq z) = f(x,V). \tag{8}$$

We refer to f as a "lossy flow" because it satisfies the usual node law for flows except for the factor $\beta < 1$.

Conversely, given a vertex set V, a probability distribution μ on $V \backslash \{z\}$, a $\beta \in (0,1)$ and a nonnegative function f on V^2 satisfying (7) and (8), define the transition kernel

$$K_z(x,y) := \frac{f(x,y)}{\beta(f(V,x) + \mu(x))}.$$

It is easy to check that this kernel corresponds to a stopped RW law (K, μ, V, z) for which the corresponding flow (6) is f.

The relevant parameters S_β, Γ_β of the stopped RW law started at a single vertex o can be expressed using the function f. Clearly:

$$S_\beta = \sum_{x \in V} f(x,z). \tag{9}$$

For the parameter Γ_β notice that for $x \in V \backslash \{z\}$, $y \in V$

$$f(x,y) = \mathcal{G}_\beta(o,x)\beta w(x,y)/w_x,$$

and therefore for $(y,x) \in \mathrm{supp}(f)$

$$\theta(x,y) := \frac{f(x,y)}{f(y,x)} \tag{10}$$

satisfies

$$\theta(x,y) = \frac{\mathcal{G}_\beta(o,x) w_y}{\mathcal{G}_\beta(o,y) w_x}. \tag{11}$$

Since $w(x,z) = w_x K(x,z)$, (3) can be written as

$$\Gamma_\beta = \sum \mathcal{G}_\beta(o,o) w_x K(x,z)/w_o, \tag{12}$$

where the sum runs over the set of vertices accessible from o without hitting z. For every such x we pick a path $\pi_x = (x_0 = o, \ldots, x_{\ell(x)} = x)$ for which $K(\pi_x) > 0$. Repeated use of equation (11) then transforms (12) to

$$\Gamma_\beta = \sum_x \theta(\pi_x) \mathcal{G}_\beta(o,x) K(x,z) = \sum_x \theta(\pi_x) f(x,z)/\beta. \tag{13}$$

1.4 Decomposition of flows

Consider a stopped random walk law (K, μ, V, z), started at a single vertex o (so μ is unit mass at o). Let f be a flow defined by this stopped random walk law (6). Fix θ as in (10). Consider the set F of nonnegative functions f_* on V^2 satisfying (8, 10) (with f replaced by f_* in both), and $\mathrm{supp}(f_*) \subset \mathrm{supp}(f)$. Note that (10) implies (7), so for each f_* it is possible to define a stopped RW law for which f_* is the corresponding lossy flow. Since the parameters Γ_β, S_β (13, 9) are linear on F, the following lemma will suffice for the proof of Proposition 3.

Lemma 4 (Decomposition of lossy flows) *Let Π be the set of simple paths from o to z. Then there exists a probability distribution α on $\Pi \cup \{o\}$ so that*

$$f = \alpha_o f_o + \sum_{\pi \in \Pi} \alpha_\pi f_\pi,$$

where $f_\pi \in F$ is supported on π and $f_0 \in F$ is supported on $(V \backslash \{z\})^2$.

For the proof of this lemma, it is useful to know which $\pi \in \Pi$ supports elements of F.

Lemma 5 *Let $\pi = (x_0 = o, x_1, \ldots, x_\ell = z)$ be a simple path. There exists $f_\pi \in F$ supported on π if and only if $\theta_i := \theta(x_i, x_{i-1}) < \beta$ for all $1 \le i \le \ell$.*

Proof. There is a unique solution f_π supported on π for the equations (8) and (10). It can be obtained inductively; set $\theta_0 := 0$, then

$$\begin{aligned} f_\pi(x_{i-1}, x_i) &= \prod_{j=1}^{i} \frac{\beta - \theta_{j-1}}{1 - \beta\theta_j}, \\ f_\pi(x_i, x_{i-1}) &= \theta_i f_\pi(x_{i-1}, x_i) \end{aligned}$$

The solution is nonnegative (equivalently, $f_\pi \in F$) if and only if $\theta_i := \theta(x_i, x_{i-1}) < \beta$ for all $1 \le i \le \ell$. □

Proof of Lemma 4. F is a closed, bounded, convex subset of a finite dimensional vector space, so it equals the closed convex hull of its extreme points (e.g. for a bit of overkill, by the Krein-Milman Theorem). Thus it suffices to prove that all extreme points of F are supported on $(V\backslash\{z\})^2$ or on simple paths. Indeed, let f_* be extreme point. Consider the directed graph on V where (x, y) is an edge iff $f_*(y, x) < \beta f_*(x, y)$ (equivalently, if $\theta(y, x) < \beta$ and $f_*(x, y) \neq 0$). Consider the set V' of vertices which are connected to z by a path directed towards z in this graph.

First suppose that $o \notin V'$. Summing the node law (8) over elements of V' yields

$$\beta f_*(V, V') - \beta f_*(V, z) = f_*(V', V). \tag{14}$$

The definition of V' implies that for $(x, y) \in (V\backslash V') \times V'$ we have $f_*(y, x) \ge \beta f_*(x, y)$; summation yields

$$f_*(V', V\backslash V') \ge \beta f_*(V\backslash V', V'). \tag{15}$$

Adding (15), the trivial inequality $f_*(V', V') \ge \beta f_*(V', V')$, and (14) yields $0 \ge \beta f_*(V, z)$ and therefore $\mathrm{supp}(f_*) \subset (V\backslash\{z\})^2$.

The second case is when $o \in V'$, so there is a directed path π satisfying the assumptions of Lemma 5 and that $f_*(x, y) > 0$ if y follows x in π. Thus there exists an $f_\pi \in F$ supported on π, and for a small $\varepsilon > 0$, the function $(1+\varepsilon)f_* - \varepsilon f_\pi$ is nonnegative hence an element of F. As f_* is an extreme point, $f_* = f_\pi$, and the proof is complete. □

1.5 Laplace domination

In this section we complete the proof of Proposition 2 and Theorem 1 outlined in Section 1.2. We will use the notation and results introduced above.

PROOF OF PROPOSITION 2. First assume that the chain is supported on a single path $\pi = (x_0 = o, x_1, \ldots, x_\ell = z)$. Define the quantity

$$s(x,y) := \frac{\beta f(x,y) - f(y,x)}{f(x,y) - \beta f(y,x)}.$$

For $1 \le i \le n$, the definition of f implies that $s(x_{i-1}, x_i) \in (0, \beta]$, and the node law (8) applied to x_i implies that

$$\beta f(x_i, x_{i-1}) - f(x_{i-1}, x_i) = f(x_{i+1}, x_i) - \beta f(x_i, x_{i+1}).$$

This makes the following product telescope:

$$s(\pi) = \frac{\beta f(x_{\ell-1}, z) - f(z, x_{\ell-1})}{f(o, x_1) - \beta f(x_1, o)}. \tag{16}$$

Since $f(z, x_{\ell-1}) = 0$, we get $s(x_{\ell-1}, z) = \beta$. The node law (8) applied to o yields $\beta(f(x_1, o) + 1) = f(o, x_1)$ so the denominator of the right hand side of (16) also equals β. Let π_{-z} denote the path π with its last vertex removed. Then

$$f(x_{\ell-1}, z) = \beta s(\pi_{-z}). \tag{17}$$

Now consider the general case. Proposition 3 allows us to assume that our stopped RW law is a convex combination of RW laws supported on simple paths $\pi \in \Pi$ and a RW law. In particular, (17), the linearity of parameters S_β, Γ_β and the expressions (9, 13) imply that

$$\begin{aligned} S_\beta &= \beta \sum_{\pi \in \Pi} \alpha_\pi s(\pi_{-z}), \\ \Gamma_\beta &= \sum_{\pi \in \Pi} \alpha_\pi (s \times \theta)(\pi_{-z}). \end{aligned}$$

It remains to bound S_β in terms of Γ_β. Let $s_{\pi,i} := s(x^\pi_{i-1}, x^\pi_i)$, where $\pi = (x^\pi_0, \ldots, x^\pi_{\ell_\pi})$. Let

$$h(s) := s\frac{1 - s\beta}{\beta - s}$$

so that $h(s(x,y)) = (s \times \theta)(x,y) = s(x,y) \times \theta(x,y)$. Then

$$\begin{aligned}
\Gamma_\beta &= \sum_\pi \alpha_\pi \prod_{i=1}^{\ell_\pi - 1} h(s_{\pi,i}) \\
&\geq \sum_\pi \alpha_\pi h \left[\left(\prod_{i=1}^{\ell_\pi-1} s_{\pi,i} \right)^{\frac{1}{\ell_\pi - 1}} \right]^{\ell_\pi - 1} \\
&\geq \sum_\pi \alpha_\pi h \left[\left(\prod_{i=1}^{\ell_\pi-1} s_{\pi,i} \right)^{1/n} \right]^{n} \\
&\geq h \left[\left(\sum_\pi \alpha_\pi \prod_{i=1}^{\ell_\pi-1} s_{\pi,i} \right)^{1/n} \right]^{n} = h\left[(S_\beta/\beta)^{1/n} \right]^n
\end{aligned}$$

The first inequality follows from Jensen's inequality and the fact that $y \mapsto \log(h(e^y))$ is convex for $y \leq \log\beta$. The second, from the fact that the function $y \mapsto h(y^{1/n})^n$ is increasing in n for $y \in [0,1]$, and that $\ell_\pi \geq n+1$. The third inequality follows from Jensen's inequality and the fact that $y \mapsto h(y^{1/n})^n$ is convex in y for $y > 0$.

Solving the above inequality for S_β, and using the fact (5) that $g := (w_z r_{oz})^{1/n} \geq \Gamma_\beta^{1/n}$ we get

$$S_\beta \leq \beta \left(\frac{g + 1 - \sqrt{(g+1)^2 - 4\beta^2 g}}{2\beta} \right)^n = \mathbf{E}\beta^{T'_{0n}+1}. \tag{18}$$

Note that T'_{0n} is the sum of n independent copies of T'_{01}. Conditioning on the first step yields $\mathbf{E}\beta^{T'_{01}} = \beta(g + (\mathbf{E}\beta^{T'_{01}})^2)/(g+1)$, and solving this equation gives the equality in (18), a standard result. Thus the inequality in (18), in terms of $-\log\beta$, is a comparison of the Laplace transforms of T_{oz} and $T'_{0n}+1$, as required. □

Proof of Theorem 1. The expected value inequality follows from differentiating the Laplace transforms at 0. For the large deviation inequality, note that

$$\mathbf{P}(T_{oz} - 1 \leq an) \leq e^{\lambda a n} \mathbf{E} e^{-\lambda(T_{oz}-1)}$$

for every $\lambda > 0$ by Markov's inequality. Replacing the Laplace transform on the right by that of T'_{0n} we get

$$\begin{aligned}
\mathbf{P}(T_{oz} - 1 \leq an) &\leq \inf_{\lambda > 0} \mathbf{E} e^{-\lambda(T'_{0n} - an)} \\
&= (\inf_{\lambda>0} \mathbf{E} e^{-\lambda(T'_{01} - a)})^n = e^{-I(a)n}.
\end{aligned}$$

For the last equality, we used the fact that the infimum over $\lambda \in \mathbb{R}$ is achieved when $\lambda > 0$; this can be checked by direct calculation.

Direct computation, or the law of large numbers, implies the expression (1) for m, and a standard computation using the Laplace transform of T'_{01} yields its large deviation rate function I given by (2). $\square$

The function $e^{-I(\cdot)}$ is concave increasing and maps $[1, m]$ to $[0, 1]$. Linear approximation at points 1, m, gives the simpler bounds, both of which hold for all $a \in [1, m]$:

$$\begin{aligned} e^{-I(a)} &\leq (g+1)(a-1), \\ e^{-I(a)} &\leq 1 - \frac{(g-1)^2}{2g}(m-a). \end{aligned} \tag{19}$$

Chapter 2

Upper bounds on speed

2.1 Introduction

Once the transience of a random walk on a graph is determined, it is natural to ask questions about its rate of escape from the starting point. This chapter studies how the linear rate of escape (speed) is related to the structure of the graph.

The size of graphs with exponential growth can be measured in many ways. Two simple measures of growth are the **lower growth**, $\underline{\mathsf{gr}}(G)$, given by the lim inf of the nth root of the total weight of edges at distance n from a fixed vertex, which we will call the root, o, and the **upper growth** $\overline{\mathsf{gr}}(G)$ which is defined similarly with lim sup. The lower growth can be expressed using the following size measure for edge sets. For an edge or vertex, let $|\cdot|$ denote its graph distance from o. For a subset of edges K, we can define a size measure which exponentially punishes edges that are far from the root:

$$\|K\|_\beta = c^{-1} \sum_K w(e)\beta^{-|e|}, \tag{1}$$

where the normalizing constant c equals the total weight on edges adjacent to o. Let ∂W denote the edge boundary of the vertex set W. Then the expression

$$\sup\{\beta \mid \inf_W \|\partial W\|_\beta > 0\},$$

where W ranges over all balls about o, equals $\underline{\mathsf{gr}}$. If we let $W \ni o$ range over all finite sets, we get the **branching number** br of the graph; if we let $W \ni o$ range over all sets from which the walk started from o exits almost surely, we get a new quantity called the **essential branching number**, eb. It is easy to see that br and eb do not depend on the choice of o. For more intuition behind br in trees, see Lyons and Peres (1999).

The **speed** of a random walk $\{X_k\}$ on a graph is given by the process $\{|X_k|/k\}$, where $|\cdot|$ denotes graph distance from the root, which is also the starting point of the walk. The **lower speed** $\underline{S}$ of the random walk can be defined as the lim inf of the speed process; when this a.s. coincides with the lim sup (the **upper speed**, $\overline{S}$), we say that asymptotic speed exists. Our main results are the following theorems.

Theorem 6 $\underline{S} \le (\mathsf{eb}-1)/(\mathsf{eb}+1) \vee 0 \quad a.s.$

Theorem 7 $\overline{S} \le (\overline{\mathsf{gr}}-1)/(\overline{\mathsf{gr}}+1) \vee 0 \quad a.s.$

Variants of these theorems have been conjectured by Benjamini and Peres (see Peres 1997) and Lyons, Pemantle and Peres (1997). Specifically, our results solve two of the questions raised in Lyons, Pemantle and Peres (1997).

An elementary argument using the strong law of large numbers shows that for both theorems equality holds for regular trees. The intuitive meaning of these bounds is that among graphs with the same essential branching number (respectively, upper growth), none admits a simple random walk that is with positive probability faster than the one on the regular tree.

Both Theorem 6 and 7 are applications of the results in Chapter 1. With some additional work (Virág 2000b), Theorem 6 can be deduced from the expected value inequality of Theorem 1. Theorem 7 needs the full power of the large deviation inequalities of Theorem 1; these inequalities also yield a shorter proof of Theorem 6 presented in Section 2.3.

Previously, Peres gave a rough upper bound for the lower speed in trees in terms of the branching number, using a percolation argument (see Häggström 1997, Peres 1997). It follows from this bound that positive lower speed implies positive branching number. Takacs (1997, 1998), and Takacs and Takacs (1998) calculated the asymptotic speed for special classes of trees, using walk-invariant measures on tree-space.

Lyons, Pemantle, and Peres (1995) computed the speed explicitly for the simple random walk on Galton-Watson family trees. For biased random walks on Galton-Watson trees, a tree-space and random walk average version of the analogue of Theorem 6 was proved by Chen (1997).

For a tree T, one can define the distance for two rays (infinite self-avoiding paths starting at the root) with n common edges as e^{-n}. Under this distance, the set of rays, ∂T, is a compact metric space with Hausdorff dimension $\log \mathsf{br}$. Consider the λ-biased random walk on T. This moves to a neighbor of its current position with odds λ for the parent and one for each child. If this walk is transient, then erasing cycles from its path gives us a random ray; the corresponding probability measure on ∂T is called harmonic measure. The dimension $d(\lambda)$ of this measure is related to what portion of the tree the random walk could potentially explore; of course, $d(\lambda) \leq \log \mathsf{br}$. In Section 2.4 we prove that

$$\underline{S} \leq \frac{e^{d(\lambda)} - \lambda}{e^{d(\lambda)} + \lambda} \quad \text{a.s.} \tag{2}$$

This was believed to be false for general trees (Lyons, Pemantle, Peres, 1997); the same paper, as well as Benjamini and Peres (1992), show through several counterexamples that many other properties, although intuitive, do not hold for general trees.

The inequality (2) can also be thought of as a lower bound for the dimension of the harmonic measure. From this perspective, our result is related to, albeit not a proof of, a conjecture of Lyons, Pemantle and Peres (1997) that the dimension of the harmonic measure on the family tree of a Galton-Watson branching process is a.s. greater than the dimension of the corresponding measure for a more greedy random walk that moves to a uniformly chosen random offspring of its current position; in short, the greedy walker sees less of the tree. Such implications of our results to Galton-Watson trees are discussed in Section 2.5.

2.2 Upper speed

This section contains a proof of Theorem 7 and a related example. The following simple lemma is needed. Recall that if o is a vertex and z is a vertex or a vertex set, then T_{oz} denotes the hitting time of z for the walk started at o.

Lemma 8 *For every $k > 0$ we have $\mathbf{P}(T_{oz} \leq k) \leq k/(w_o r_{oz})$.*

PROOF. Let R denote the number of hits to o in the time interval $[0, T_{oz}]$. Then R is a geometric random variable with expected value $w_o r_{oz}$. Thus

$$\mathbf{P}(T_{oz} \leq k) \leq \mathbf{P}(R \leq k) = 1 - (1 - (w_o r_{oz})^{-1})^{\lfloor k \rfloor} \leq k/(w_o r_{oz}). \quad \square$$

Proposition 9 *For $1 < g' < g$, there exist $c_1, c_2 > 0$ so that the following holds. If in a weighted graph $\mathrm{dist}(o, z) \geq n$ for some $n > 0$ and $w_z/w_o \leq g'^n$, then with*

$$m(g) = (g+1)/(g-1) \tag{3}$$

we have

$$\mathbf{P}[T_{oz} \leq m(g)n] \leq c_1 e^{-c_2 n}.$$

PROOF. Let $g'' < g'''$, both in the interval (g', g). We have $w_z/w_o < g'^n$, so unless $r_{oz} w_o > (g''/g')^n$, in which case

$$\mathbf{P}(T_{oz} \leq m(g)n) \leq m(g)n(g''/g')^n \tag{4}$$

by Lemma 8, we must have $w_z r_{oz} < g''^n$ and so for large n satisfying $m(g)n < m(g''')(n+1)$

$$\mathbf{P}(T_{oz} \leq m(g)n) \leq e^{-I_{g''}(m(g'''))(n+1)} \tag{5}$$

by Theorem 1. The statement of the proposition follows from (4) and (5). $\square$

We are ready to prove of Theorem 7.

PROOF OF THEOREM 7. Let $g > \overline{\mathsf{gr}} \wedge 1$ be arbitrary, and let T_n, w_n denote the hitting time and the total weight of the set of vertices at distance n from o, respectively. Then

$$\limsup \mathrm{dist}(o, X_k)/k = \limsup n/T_n. \tag{6}$$

For all large n we have $w_n/w_o < g^n$, so by Proposition 9 each event $T_n \leq an$ has probability at most $c_1 e^{-c_2 n}$. By the Borel-Cantelli Lemma only finitely many of these events happen. Thus (6) is at most $m(g)^{-1}$, and since $g > \overline{\mathsf{gr}} \wedge 1$ was arbitrary, the Theorem follows. $\square$

Example 10 Let $g \geq 2$ be an integer, and consider the graph of the nonnegative integers with g-ary trees of depth d_i attached at vertex i for every i. If d_i increases fast enough, then by the time the walk started from 0 visits the leaves of the tree at d_i, its speed will be nearly as high as the speed of the walk on the g-ary tree, and the upper growth of this graph is just g. This gives an example of a recurrent graph for which equality is achieved in Theorem 7.

2.3 Lower speed

In order to bound the lim inf speed in terms of the essential branching number we need a few simple lemmas.

Lemma 11 (Doob transform) *Let z be a subset of vertices in a weighted graph $G = (V, w)$, and let τ be the lifetime of the random walk $\{X_k\}$ started at $o \in W$ and killed exits W. Let A be an event $\{X_k \notin V, 0 \leq k \leq t\}$. Let G' be the re-weighting of G determined by the edge weights*

$$w'(u,v) = w(u,v)\mathbf{P}_u A \mathbf{P}_v A,$$

and let $\{X'_k\}$ be random walk on G' started at o Then $\{X_k\}_{0 \leq k \leq T_{oz}}$ given A and $\{X'_k\}_{0 \leq k \leq T'_{oz}}$ have the same distribution. Moreover, we have

$$\|\partial W'\|_g \leq (\mathbf{P}_o A)^{-2} \|\partial W\|_g.$$

Proof. By summing over paths it is easily checked that $\{X_k\}$ conditioned on A is a Markov chain with stationary transition probabilities given by

$$\mathbf{P}(X_k = v | X_{k-1} = u, A) = \mathbf{P}_u(X_1 = v) \frac{\mathbf{P}_v A}{\mathbf{P}_u A}. \tag{7}$$

But $\{X'_k\}$ has exactly these transition probabilities. If E_o is the set of vertices adjacent to o, then $w'(E_o)$ can be written as

$$\sum_{v \sim o} w(o,v)\mathbf{P}_o A \mathbf{P}_v A = \mathbf{P}_o A \sum_{v \sim o} w(E_o)\mathbf{P}_o(X_1 = v)\mathbf{P}_v A = (\mathbf{P}_o A)^2 w(E_o).$$

The second statement follows from substituting this into (1) as a normalizing constant. □

Proposition 12 *Let $1 < g' < g$, then there exist $c_1, c_2 > 0$ so that the following holds. If in a weighted graph $\mathrm{dist}(o,z) \geq n$ for some $n > 0$ and $\|\partial W\|_{g'} \leq 1$ then*

$$\mathbf{P}[T_{oz} \leq m(g)\mathrm{dist}(o, X_{T_{oz}})] \leq c_1 e^{-c_2 n}. \tag{8}$$

PROOF. Pick $g'' \in (g', g)$. In order to get a bound on the exit speed, we will have to condition on the distance of the $X_{T_{oz}}$ from the root o. Let $B_\ell := \{|X_{T_{oz}}| = \ell\}$. Consider the B_ℓ which have relatively high probability: $L = \{\ell : \ \mathbf{P}B_\ell > (g'/g'')^{\ell/2}\}$. Let B denote the union of B_ℓ for $\ell \in L$. Note that

$$\mathbf{P}B^c \leq \sum_{\ell=n}^{\infty} (g'/g'')^\ell. \tag{9}$$

Decomposing the event A given in (8) we get

$$\mathbf{P}A \leq \sum_{\ell \in L} \mathbf{P}(A|B_\ell)\mathbf{P}(B_\ell) + \mathbf{P}B^c \leq \inf_{\ell \in L} \mathbf{P}(A|B_\ell)\mathbf{P}B + \mathbf{P}B^c \tag{10}$$

Let V^ℓ denote the Doob transform of V with respect to the event B_ℓ. Given B_ℓ, each edge adjacent to $X_{T_{oz}}$ has graph distance at most ℓ from o, so for $\ell \in L$, Lemma 11 implies that

$$(g'/g'')^\ell g'^{-\ell} \|\partial V^\ell\|_1 \leq (g'/g'')^\ell \|\partial V^\ell\|_{g'} \leq \|\partial V\|_{g'} \leq 1.$$

Thus $\|\partial V^\ell\|_1 \leq g''^\ell$, and Proposition 9 applied to V^ℓ implies

$$\mathbf{P}(A|B_\ell) \leq c_1' e^{-c_2' n}. \tag{11}$$

With a large c_1 to take care of small n, (8) follows from (9), (10) and (11). □

The results presented here are relatively simple consequences of the positive probability bounds above.

Theorem 13 *The lower speed $\underline{S} = \liminf |X_k|/k$ of the random walk on an infinite weighted graph and the essential branching number* eb *satisfy $\underline{S} \leq (\mathsf{eb} - 1)/(\mathsf{eb} + 1) \vee 0$ a.s.*

Remark. Since $\mathsf{eb} \le \mathsf{br} \le \underline{\mathsf{gr}}$, we can replace eb in Theorem 13 by any of these quantities. Note that when $\mathsf{br} \le 1$, this statement means that $\underline{S} = 0$. A nice proof of $\mathsf{br} = 1 \Rightarrow \underline{S} = 0$ in trees using percolation is due to Peres; see Häggström (1997) and Peres (1997).

PROOF. Let $g > g' > \mathsf{eb} \wedge 1$. Then there are sets W from which the walk exits almost surely with arbitrary small size $\|\partial W\|_{g'}$. In particular, given n, we can find such a set W_i satisfying $\|\partial W_n\|_{g'} \le 1$ and $\mathrm{dist}(o, \partial W_n) > i$. Let T_n be the hitting time of the complement of W_n.

By Proposition 12, the probability that the speed at time T_n is greater than $m(g)$ is bounded above by $c_1 e^{-c_2 n}$, which is summable. Therefore

$$\liminf |X_k|/k \le \limsup n/T_n \le m(g)^{-1}$$

and since $g > \mathsf{eb} \wedge 1$ was arbitrary, the theorem follows. □

2.4 Trees and harmonic measure

Let T be a weighted infinite tree satisfying

$$w(e)/w(e^*) \le \gamma \tag{12}$$

for some positive γ and all edges e, where e^* is the edge adjacent to e closest to the root. A **ray** in T is an infinite self-avoiding path starting from the root. The quantities br, eb are related to dimension properties of the set of rays of T called the boundary, ∂T. For two rays φ, ψ, we denote the edge farthest from the root in their intersection by $\varphi \wedge \psi$. For trees satisfying (12), the distance

$$\mathrm{distance}(\varphi, \psi) = w(\varphi \wedge \psi)\gamma^{-|\varphi\wedge\psi|} \tag{13}$$

can be easily checked to satisfy the triangle inequality. Moreover, the boundary ∂T under this distance is compact. Also, any open or closed ball about a ray φ is given by the set of all rays that eventually stay in a descendant subtree of a vertex in φ.

Using this fact, it is easy to check that the Hausdorff dimension of the boundary satisfies $\gamma^{\dim_\gamma \partial T} = \mathsf{br}(T)$.

If the random walk is transient on T, then its loop-erased path is a random element of ∂T. The corresponding measure μ is called the harmonic measure on ∂T. The Hausdorff dimension $\dim_\gamma \mu$ of μ is defined as the infimum of Hausdorff dimensions of Borel sets with full μ-measure. It satisfies

$$\gamma^{\dim_\gamma \mu} \geq \mathsf{eb}(T). \tag{14}$$

To check this, note that if $d > \dim_\gamma \mu$, then there is a set of full measure with dimension less than d; there are covers where the sum of the dth powers of the diameters is arbitrarily small, so there are sets W which the walk exits almost surely with $\|\partial W\|_{\gamma^d}$ arbitrarily small.

The inequality (14) can be strict. Consider an infinite ternary tree, with binary trees of depth $h(|v|)$ attached to each vertex for some rapidly growing function h. Consider the vertex sets W_i whose boundary of all the leaves of the binary trees starting at level $n_1, n_2, \ldots$ for some infinite sequence. For any $g > 2$, we can find a sequence $n_1, n_2, \ldots$ so that the corresponding set has arbitrarily small boundary size $\|\partial W\|_g$. Hence such a tree has $3 = \gamma^{\dim_\gamma \mu} > \mathsf{eb}(T) = 2$. The harmonic measure ignores detours of the walk—the essential branching number takes them into account.

The λ-**biased random walk** in a tree moves to a random one of the neighbors of its current position, with odds λ for the parent and 1 for each child. We then have the following corollary to Theorem 13.

Corollary 14 *Consider a transient λ-biased random walk on an infinite unweighted tree T. Let $d(\lambda)$ denote the dimension of the harmonic measure on ∂T with respect to this walk and the metric $e^{-|a \wedge b|}$, and let $\underline{S}_\lambda$ denote the lower speed. Then*

$$\underline{S}_\lambda \leq \frac{e^{d(\lambda)} - \lambda}{e^{d(\lambda)} + \lambda} \vee 0 \quad a.s.$$

Proof. The λ-biased random walk on T has the same law as the one in the weighted tree T' that has the same graph structure as T, but with edge weights $w(e) = \lambda^{-|e|}$. Denote the harmonic measure by μ, and let $\gamma > \lambda^{-1}$. Denote $\dim_\gamma$, $\dim'_\gamma$

Hausdorff dimension with respect to the distance (13) in T and T', respectively. Then from the definitions we get $\dim'_\gamma \mu = \log_\gamma \lambda^{-1} + \dim_\gamma \mu$, and that $d(\lambda) = \dim_e \mu = \ln\gamma \dim_\gamma \mu$. Combining these with (14), we get $\mathsf{eb}(T') \leq e^{d(\lambda)}/\lambda$. The result now follows from Theorem 13 applied to T'. □

Remark. Lyons, Pemantle and Peres (1997) show that, for transient biased walks, $e^{d(\lambda)} \geq \lambda$, so we can remove the "$\vee 0$" from the statement of Corollary 14.

2.5 Galton-Watson trees

Let T be the family tree of a Galton-Watson branching process with offspring distribution Z, and suppose that $Z \geq 1$, that is, each parent has at least one child, and that Z is non-constant. Consider a transient λ-biased random walk on T. It is known that the asymptotic speed s_λ exists and is constant a.s.

Let $\dim \partial T$ denote the Hausdorff dimension of the boundary. Hawkes (1981) and Lyons (1990) showed that $\dim \partial T = \log \mathbf{E} Z$ a.s. The dimension $d(\lambda)$ of the harmonic measure is also known to be constant for each λ a.s.

Transience implies $\log \lambda \leq \dim \partial T$ (Lyons, 1990). Lyons, Pemantle and Peres (1996) proved that when $\log \lambda \neq \dim \partial T$, the dimension $d(\lambda)$ is strictly less than $\dim \partial T$ if $\mathbf{E} Z \log Z < \infty$. A conjecture of the same authors (1997) follows from Corollary 14 (note the strict inequality):

Corollary 15 *For the asymptotic speed s_λ of the λ-biased random walk on Galton-Watson trees with non-constant offspring distribution Z satisfying $\mathbf{E} Z > \lambda$, we have:*

$$s_\lambda < \frac{\mathbf{E} Z - \lambda}{\mathbf{E} Z + \lambda}.$$

The dimension of the harmonic measure $d(\lambda)$ gauges the size of the subtree of T the random walk can potentially explore. Heuristically, as λ increases, the random walk tends to backtrack more, and thus explores more of the tree, so one expects that $d(\lambda)$ is an increasing function of λ. Counterexamples to this heuristic are given for general deterministic trees and for family trees of multi-type branching processes in Lyons, Pemantle, Peres (1997). However, for simple Galton-Watson trees it is still unknown whether $d(\lambda)$ is monotone. In the case $\lambda = 0$ the walk always moves away from the root; the resulting harmonic measure is called "visibility measure". Lyons, Pemantle, Peres (1995) showed that $d(0) = \mathbf{E}\log Z$ a.s.

Corollary 14 can be used to give a lower bound for general $d(\lambda)$. We have

$$\frac{1/\mathbf{E}Z^{-1} - \lambda}{1/\mathbf{E}Z^{-1} + \lambda} < s_\lambda \leq \frac{e^{d(\lambda)} - \lambda}{e^{d(\lambda)} + \lambda},$$

where the first inequality is due to Chen (1997). For $\lambda = 1$, Lyons, Pemantle and Peres (1996) proved that $s_1 = \mathbf{E}[(Z-1)/(Z+1)]$ a.s. Putting these results together, we get:

Corollary 16 *With m as in (3), the dimension $d(\lambda)$ of the harmonic measure of the biased random walk on a Galton-Watson tree satisfies*

$$d(\lambda) > -\log \mathbf{E}(Z^{-1}), \qquad d(1) \geq \log m\left(\mathbf{E}(m(Z)^{-1})\right).$$

To summarize what is known about the dimension $d(\lambda)$, note the following consequence of Jensen's inequality:

$$-\log \mathbf{E}(Z^{-1}) \leq \log m\left(\mathbf{E}(m(Z)^{-1})\right) \leq \mathbf{E}\log Z \leq \log \mathbf{E}Z. \tag{15}$$

The first and second expressions are the lower bounds for $d(\lambda)$ for general λ and $\lambda = 1$. The third is the exact value of $d(0)$, the last is the dimension of the boundary. Respectively, Corollary 16 is not sufficient to establish monotonicity properties; however, it provides the first known lower bound for $d(\lambda)$ (apart from $d(\lambda) \geq \log\lambda$, see the remark to Corollary 14).

Chapter 3

Anchored expansion and random walk

3.1 Introduction

Anchored expansion was introduced by Benjamini, Lyons, and Schramm (1999) as a non-uniform version of the strong isoperimetric inequality, after Thomassen (1992) used more general "anchored" isoperimetric inequalities to give sufficient conditions for the transience of random walks on graphs. Define the **volume** $|\cdot|$ of an edge or vertex set as the sum of the weights over the set, and define the **edge boundary** ∂S of a vertex set S as the set of edges with one vertex inside S and one outside. The **strong isoperimetric inequality** with constant i, perhaps the simplest isoperimetric inequality, states that

$$|\partial S| \geq i|S| \tag{1}$$

for all vertex sets S, and the **Cheeger constant** of a graph is the maximal i for which this inequality holds. The implications of a positive Cheeger constant are strong; the Markov kernel on a graph with positive Cheeger constant has spectral radius less than 1 (Cheeger (1970), Dodziuk (1984), Mohar (1988); these two conditions are in fact equivalent) and as a result, if the graph does not grow faster than exponentially, the random walk escapes at a positive speed, (i.e. linear rate). As it is clear from the definition, having a positive Cheeger constant is a rather fragile property. Random perturbations, such as Bernoulli percolation on an (unweighted) graph, even with a very high p, or a geometric stretching of edges, destroy it; see Benjamini, Lyons and Schramm (1999) for results on the stability of graph properties under random perturbations.

A more stable condition, which we call ***i*-anchored expansion**, requires (1) to hold with possibly some exceptions provided that every vertex is contained in only finitely many connected exceptional S (of course, there need not be a uniform bound on the number of such exceptions). We call the supremum of i for which a graph G has i-anchored expansion the **anchored expansion constant** $\mathbf{i}(G)$; if this constant is greater than 0, then we say that G has the **anchored expansion property**. If G is connected with edge weights bounded from below (in particular, if G is a connected, unweighted graph), then this is equivalent to a version of the original definition,

$$\mathbf{i}(G) = \liminf_{S \ni v} \frac{|\partial S|}{|S|},$$

where S ranges over all connected vertex sets containing a fixed vertex v; the lim inf is applied to the *set* of values obtained. This definition does not depend on the choice of the fixed vertex, and it explains the name "anchored expansion". The only difference from the original definition (given for the unweighted case) is that here the volume of a vertex set is defined as the sum of degrees rather than the number of vertices.

Benjamini, Lyons and Schramm (1999) conjectured that in an unweighted graph with bounded degree and anchored expansion, the random walk has positive lim inf speed with positive probability. The main goal of this chapter is to prove this conjecture in a slightly stronger form. Say a weighted graph has **w^*-bounded geometry** if all positive edge weights are at least 1 and all vertex weights are at most w^*, and let $|X_n|$ denote the graph distance between the random walker and the initial vertex at time n.

Theorem 17 *There exists $c > 0$ so that the random walk on a weighted graph with w^*-bounded geometry satisfies* $\liminf |X_n|/n \geq c\mathbf{i}(G)^7 {w^*}^{-3}$ *a.s.*

This theorem gives a geometric explanation for positive speed in certain graphs, such as infinite components of p-Bernoulli percolation on graphs with positive Cheeger constant for high p, geometric edge-stretchings of such graphs, or supercritical Galton-Watson trees. For these graphs, Chen and Peres (1998), inspired by questions of Benjamini, Lyons, and Schramm (1999), proved the anchored expansion property. It is an open question (see Häggström, Schonmann and Steif (1999)) whether infinite clusters of Bernoulli percolation on a transitive graph have the anchored expansion property. The same authors prove that if G is a transitive graph and there exists an automorphism-invariant percolation on G where all infinite components have the anchored expansion property, then G has positive Cheeger constant.

Since an exponential heat kernel bound is equivalent to having a positive Cheeger constant, one cannot hope that anchored expansion would imply such a strong bound. The following theorem gives a sub-exponential bound, which is strongest in the sense that the $n^{1/3}$ in the exponent cannot be improved.

Theorem 18 *Let G be a weighted graph with the anchored expansion property and w^*-bounded geometry. Let $\alpha := \mathbf{i}(G)^2(w^{*2}/2)^{-1/3}/9$. For every vertex x there is an N so that*

$$p^n(x,y) < e^{-\alpha n^{1/3}} \quad \textit{for all } n > N, \ y \in V.$$

Varopoulos (1991) showed that such a bound holds in Cayley graphs of any finitely generated group of exponential growth (see Hebisch and Saloff-Coste (1993) for a more general statement and a simpler proof). Such a group can be amenable (e.g. the lamplighter group G_1), in which case it is an example where such decay holds but anchored expansion does not.

Section 3.3 examines the geometry of graphs with i-anchored expansion, and shows that they are built from a graph with Cheeger constant at least i and "islands", each of which is finite but whose size is not necessarily bounded by a constant (a binary tree with "pipes" of increasing length attached to a scarce set of vertices is a graph with anchored expansion and unbounded "islands"). Section 3.4 proves some properties of random walk on such graphs. Using these results, in Section 3.5 we prove that the random walk has positive speed, and in Section 3.6 we establish the heat kernel bound.

3.2 Notation

The concept of **the random walk on a weighted graph** $G = (V, w)$ is just a geometric representation of a countable, reversible Markov chain with transition probabilities

$$p(u, v) = w(u, v)/w_u.$$

Conversely, if we have a reversible Markov chain with transition probabilities $p(u, v)$ and stationary measure w_v we get a weighted graph by the above formula. We will use the notation P for the transition probability matrix of the walk. We will usually denote the walk itself by $\{X_n\}$, and $\mathbf{P}_v$, $\mathbf{E}_v$ will mean probability and expectation with respect to the walk started at vertex v.

We will consider the Hilbert space $L^2(V, w)$ of functions, equipped with the inner product and norm

$$(f, g) = \sum_{v \in V} f(v)g(v)w_v, \qquad \|f\| = (f, f)^{\frac{1}{2}}.$$

For an operator P on $L^2(V, w)$, we will use the norm $\|P\| = \sup \|Pf\|/\|f\|$. The **Markov kernel** P of a random walk on a weigthed graph is the operator on $L^2(V, w)$ defined by $(Pf)(v) := \mathbf{E}_v f(X_1)$.

We will use the standard notation $\boldsymbol{\delta}_x$, for unit mass at x (formally, $\boldsymbol{\delta}_x$ is an element of the dual of $L^2(V, w)$), and $\mathbf{1}_S$ for the indicator (characteristic) function of the set S. The volume of a vertex set S can be written as $|S| = \sum_{v \in S} w_v = \|\mathbf{1}_S\|^2$. The **number of vertices** of S will be denoted $\#S$.

The **induced subgraph** of $S \subset V$ is the graph G with vertex set S and edge set given by $\{(u, v) \in E : u, v \in S\}$, and we call S **connected** if its induced subgraph is connected. We will often write $G \setminus S$ for the induced subgraph of $V \setminus S$. The **inner vertex boundary** of S is the set of vertices in S with a neighbor outside S, and the **outer vertex boundary** of S the set of vertices outside S with a neighbor in S.

By a **path** of **length** n we mean a subgraph with vertex set $(v_0, \ldots, v_n)$ and edge set consisting of edges between consecutive v-s. The **graph distance** between two vertices in G is the length of the shortest path with endpoints given by the two vertices. The notation $|v|$ for a vertex denotes the graph distance between v and some declared fixed vertex; this vertex will usually be the starting point of the random walk we study. The **lim inf speed** of the walk is defined as $\liminf |X_n|/n$.

We will use c, c_2, c_3 to denote constants whose values might change from one expression to another.

3.3 Geometry of graphs with anchored expansion

Let $G = (V, w)$ be a weighted graph. For $i < 1$, define the i-**isolation** of a finite-volume vertex set S of G by

$$\Delta_i S = i|S| - |\partial S|.$$

A vertex set S with positive i-isolation will be called i-**isolated**. We will call a vertex set S satisfying $\Delta_i S > \Delta_i A$ for every subset $A \neq S$ of S an i-**isolated core**. Since A can be the empty set, an isolated core must be either empty or isolated. A nice property of i-isolated cores is given in the following lemma.

Lemma 19 *Let A be a vertex set and let S be an i-isolated core which is not a subset of A. Then $\Delta_i(A \cup S) > \Delta_i A$.*

PROOF. Note that if B and C are disjoint vertex sets then

$$\Delta_i(B \cup C) = \Delta_i B + \Delta_i C + 2|\partial B \cap \partial C|. \tag{2}$$

The factor 2 in the above expression appears since common boundary edges of B and C are not boundary edges of their union. Then $\Delta_i(A \cup S) = \Delta_i(A \setminus S) + \Delta_i S + 2|\partial(A \setminus S) \cap \partial S|$. The hypothesis can be used to bound the second term. The last term equals twice the total weight on edges with one endpoint in $A \setminus S$ and the other endpoint in S; this does not increase if we require the latter endpoint to be in a subset of S. Therefore

$$\Delta_i(A \cup S) > \Delta_i(A \setminus S) + \Delta_i(A \cap S) + 2|\partial(A \setminus S) \cap \partial(A \cap S)| = \Delta_i A. \quad \square$$

Corollary 20 *The union of finitely many i-isolated cores is an i-isolated core.*

PROOF. It suffices to prove this for two i-isolated cores S, S'. Let $A \subset S \cup S'$. Two applications of the lemma imply $\Delta_i A \leq \Delta_i(A \cup S) \leq \Delta_i(A \cup S \cup S')$, and one inequality is strict unless $A = S \cup S'$. □

Let A_i denote the union of all i-isolated cores in G. It follows from the definitions and Corollary 20 that if G has i-anchored expansion then every connected component of A_i is a *finite* union of isolated cores, hence an i-isolated core.

If G has i-anchored expansion, then the set A_i has the remarkable property that $G \setminus A_i$ is a graph with Cheeger constant at least i. Indeed, let S be a finite subset of $V \setminus A_i$, and let C be a (possibly empty) i-isolated core containing all vertices adjacent to S in A_i. A (possibly empty) minimal subset B of $C \cup S$ satisfying $\Delta_i B \geq \Delta_i(C \cup S)$ must be an i-isolated core, hence $B \subset A_i$, and thus $B \subseteq C$. Since C is an i-isolated core, we get

$$\Delta_i(C \cup S) \leq \Delta_i B \leq \Delta_i C. \tag{3}$$

Let $\Delta_i^{G \setminus A_i}$ denote i-isolation of vertex sets in the graph $G \setminus A_i$. When A_i is removed from G, the volumes of both S and ∂S decrease by $|\partial C \cap \partial S|$. Thus we get $\Delta_i^{G \setminus A_i} S = \Delta_i S + (1-i)|\partial C \cap \partial S|$. Expressing $\Delta_i S$ by (2) gives

$$\Delta_i^{G \setminus A_i} S = \Delta_i(C \cup S) - \Delta_i C - 2|\partial C \cap \partial S| + (1-i)|\partial C \cap \partial S|.$$

This is at most 0 by (3), and we get the required isoperimetric inequality. Thus we have shown

Proposition 21 *Every graph with i-anchored expansion contains a subgraph with Cheeger constant at least i.*

Note that $G \setminus A_i$ is an isomorphism-invariant function of the graph G. If, for example, G' is a transitive graph and $G \subset G'$ is a random subgraph whose law is invariant under a group of automorphisms of G', then the law of $G \setminus A_i$ is also invariant under this group.

We will call the connected components of A_i (i-)**islands**, and $G \setminus A_i$ the **oceans** (plural since $G \setminus A_i$ is not always connected). If $i' < i$, then we have $\Delta_i S = (i - i')|S| + \Delta_{i'} S$, so i'-isolated sets are also i-isolated, and if $A \subset S$ and $\Delta_{i'} A < \Delta_{i'} S$, then $\Delta_i A < \Delta_i S$. In particular, i'-isolated cores are i-isolated cores as well, giving $A_{i'} \subset A_i$. Thus decreasing i has the effect of global warming: it raises the level of the oceans. The following lemma gives an upper bound on how much the level needs to be raised to sink certain islands.

Lemma 22 *Let G be a connected graph with i-anchored expansion and edge weights bounded below by 1. Let S be a union of islands, each having volume at most i'^{-1} for some $i' > 0$. Then $S \subset V \setminus A_{i'}$.*

PROOF. Any i-island in S has boundary volume at least 1 and positive i-isolation, thus volume greater than i^{-1}. This gives $i' < i$. Similarly, each i'-island has volume greater than i'^{-1}, so no i'-island is a subset of an island in S. But since $A_{i'} \subset A_i$, this implies that $A_{i'} \cap S = \emptyset$. □

Let N_m denote the mth positive time the walk is in $G \setminus A_i$. The process $\{X_{N_m}\}$ (often called the induced Markov chain on $G \setminus A_i$) is also a reversible Markov chain, that is a random walk on a weighted graph G_i. If G is connected, then so is G_i. The vertex set of G_i is given by $V \setminus A_i$, and its edge weight function satisfies

$$w_i((u,v)) = w_u \mathbf{P}_u(X_{N_1} = v).$$

Clearly, for $u, v \in V \setminus A_i$ we have $w_i(u,v) = w(u,v)$ unless u and v are both on the outer vertex boundary of the same island in G. It is also clear that for $v \in V(G_i)$, we have $w_i(v) = w(v)$. The reversibility of the walk on G implies that w_i is a symmetric function on the edges.

The graph G_i has the same vertex set as $G \setminus A_i$, but its edge and vertex weights are greater or equal. We now show that G_i also has Cheeger constant at least i. To see this, let S be a finite subset of $V \setminus A_i$ and follow the argument for $G \setminus A_i$ to get (3). Let superscript G_i on volume or i-isolation denote these quantities measured in the base graph G_i. Since we have $|S|^{G_i} = |S|$ and $|\partial S|^{G_i} \geq |\partial S| - |\partial C \cap \partial S|$, it follows by (2) and (3) that

$$\Delta_i^{G_i} S \leq \Delta_i S + |\partial C \cap \partial S| = \Delta_i(C \cup S) - \Delta_i C - |\partial C \cap \partial S| \leq 0.$$

The upcoming analysis of random walks will need a rigorous formulation of the idea that large islands cannot be very close to each other. One could expect islands to have a neighborhood, whose radius depends on the size of the island, within which there are no other islands; or if this cannot be achieved, then at least one could group nearby islands together to get such a configuration. This is too optimistic as said, but Proposition 24 has a similar decomposition, for which we first have to introduce some tools.

A **bridge structure** interconnecting a vertex set $S \subset V(G)$ is a set of vertices B so that $B \cup S$ is a connected. A **bridge** connecting two vertex sets $S_1, S_2 \subset V(G)$ is a vertex set B so that the vertex set $B \cup S_1 \cup S_2$ has a connected component intersecting both S_1 and S_2. Define the ***i*-length** of a bridge B as the number of its vertices in the ocean, $\#(B \setminus A_i)$. For a vertex set S and a vertex $v \notin S$ let $\mathrm{dist}_i(v, S)$ be equal 1 plus the length of the shortest bridge connecting $\{v\}$ and S; for $v \in S$, let $\mathrm{dist}_i(v, S) := 0$.

Lemma 23 *Let G be a weighted graph with w^*-bounded geometry and $\mathbf{i}(G) > i > 0$ for some i. Let $\mathcal{R}$ be a set whose elements are unions of i-islands, and let v be a vertex. Suppose that for each $R \in \mathcal{R}$, there exists a bridge structure B which interconnects $R \cup \{v\}$ and satisfies*

$$w^* \#(B \cup \{v\} \setminus A_i)/|R| \leq \mathbf{i}(G) - i. \tag{4}$$

Then $\mathcal{R}$ is finite.

Proof. For $R \in \mathcal{R}$, let S denote $(B \cup \{v\}) \setminus A_i$, and let A denote the union of islands intersecting $B \cup R \cup \{v\}$. Then

$$|\partial(A \cup S)| \leq |\partial A| + |\partial S| \leq i|A| + |S|.$$

The bound on the first term of the second inequality holds since A is a union of islands. By (4) we have $|S|/|A| \leq \mathbf{i}(G) - i$. Therefore, using that $i < 1$, we get

$$\frac{|\partial(A \cup S)|}{|A \cup S|} \leq \frac{i|A| + |S|}{|A| + |S|} = \frac{i + |S|/|A|}{1 + |S|/|A|} \leq \frac{\mathbf{i}(G)}{1 + \mathbf{i}(G) - i} < \mathbf{i}(G).$$

By the anchored expansion property there are only finitely many such sets $A \cup S$ containing v. The lemma follows. □

Proposition 24 *Let G be a graph with $\mathbf{i}(G) > 0$ and w^*-bounded geometry. Let*

$$0 < i \leq \tfrac{2}{3}\mathbf{i}(G), \quad r(\ell) := a2^\ell/\ell^2, \quad a := \tfrac{3}{2}\pi^{-2} i w^{*-1}.$$

For each positive integer ℓ there is a (possibly empty) collection Ξ_ℓ of vertex sets C, which we call **level ℓ countries***, so that the following hold:*

- *For each ℓ and $C \in \Xi_\ell$, the set $C \cap A_i$ is a union of i-islands, and is called the* **land** *of the country C. Its volume satisfies $|C \cap A_i| \in [2^{\ell-1}, 2^\ell)$.*
- *For each ℓ and $C \in \Xi_\ell$, $C \setminus A_i = \{v \in V(G_i) : \mathrm{dist}_i(v, C \cap A_i) \leq r(\ell)\}$, and this set is called the* **waters** *of the country C.*
- *Any two countries at the same level are disjoint.*
- *Every i-island is a subset of some country.*
- *Each vertex of G is contained in at most finitely many countries.*

Proof. We start by constructing **regions** R, which are islands or unions of islands, together with bridge structures $B(R)$ connecting these islands if they are disjoint. First, for each $\ell \geq 1$, label each i-island with volume in $[2^{\ell-1}, 2^\ell)$ as a level ℓ region, and for these regions R, set $B(R) = \emptyset$.

Define the **waters** of a level ℓ region R as $\{v \in V(G_i) : \mathrm{dist}_i(v, R) \leq r(\ell)\}$. Then, for $\ell = 1, 2, \ldots$ (in this order), consider a maximal matching of pairs of level ℓ regions whose waters intersect, and label the union R of each matched pair (R_1, R_2) of regions a level $\ell + 1$ region. Set $B(R)$ to be the union of $B(R_1)$, $B(R_2)$ and a shortest (minimal length) bridge connecting R_1 and R_2.

For every level ℓ region R and vertex v in R or in its waters, consider the bridge structure $B(v, R)$ given by the union of $B(R)$ and a shortest bridge connecting $\{v\}$ and R. We have

$$\#(B(v,R) \cup \{v\} \setminus A_i) \leq r(\ell) + \sum_{n=1}^{\ell-1} 2^{\ell-1-n} \cdot 2r(n) < a2^\ell \sum_{n=1}^{\infty} n^{-2} = \frac{i}{w^*} 2^{\ell-2}.$$

In the second expression the first term is an upper bound on the length of a shortest bridge connecting $\{v\}$ and R plus 1. The first factor in the sum is an upper bound on the number of pairs of level n regions contained in R; the second factor is an upper bound of the length of a shortest bridge connecting such a pair.

Since then $w^* \#(B(v,R) \cup \{v\} \setminus A_i)/|R| < i/2 \leq \mathbf{i}(G) - i$, it follows by Lemma 23 that each vertex is contained in only finitely many regions or their waters. Therefore, every island is contained in a maximal region, that is a region which is not contained in any other regions. Call the union of a maximal region and its waters a country of level the level of the maximal region. Call the region itself the land of the country. This construction clearly satisfies the properties claimed in the proposition. □

3.4 Random walk and anchored expansion

Let $\{X_n\}$ be the random walk on a graph G with i-anchored expansion and w^*-bounded geometry. Our strategy for the analysis of this walk will be to handle the time spent in the oceans and in the islands separately. Let N_m be the m-th positive time when $\{X_n\}$ visits a vertex in $G \setminus A_i$. We have seen that $\{X_{N_m}\}$ is the random walk on the graph G_i, which has Cheeger constant at least i. First, we show that

$$\liminf |X_{N_m}|/m \geq \frac{|\log(1-i^2)|}{\log w^*} > \frac{i^2}{\log w^*} \quad \text{a.s.} \tag{5}$$

For this, we first quote a version of the classical result of Cheeger (1970), Dodziuk (1984) and Mohar (1988), to be found, for example, in Lyons and Peres (1998).

Proposition 25 *Let P be the Markov kernel of the random walk on a weighted graph with Cheeger constant at least i. Then $\|P\| \leq (1-i^2)^{1/2} \leq (1-i^2/2)$.*

This, together with the following lemma implies (5).

Lemma 26 *Let G be a weighted graph with $\|P\| < 1$, and let f be a nonnegative vertex function so that*

$$g := \limsup |f^{-1}([0,n])|^{1/n} < \infty.$$

Then $\liminf f(X_n)/n \geq -2\log\|P\|/\log g$ *a.s.*

If we set $f(v) := |v|$ in G (this might be different from $|v|$ measured in G_i), then the bounded geometry property implies that $g \leq w^*$, and the lemma applied to the walk X_{N_m} on G_i implies (5). In a similar fashion, we get the bound $i^2/\log g$ for the lim inf speed of random walks in graphs with Cheeger constant at least i and exponential growth rate at most g.

PROOF. For a small $\varepsilon > 0$, let $a := -2\log\|P\|/\log(g+\varepsilon) - \varepsilon$. We have

$$\begin{aligned} \mathbf{P}_x[f(X_n) \leq an] &= \boldsymbol{\delta}_x P^n \mathbf{1}_{f^{-1}([0,an])} = w_x^{-1}(\mathbf{1}_x, P^n \mathbf{1}_{f^{-1}([0,an])}) \\ &\leq w_x^{-1}\|P^n\|\|\mathbf{1}_{f^{-1}([0,an])}\| \end{aligned}$$

For sufficiently large n, this is bounded above by $w_x^{-1}\|P\|^n(g+\varepsilon)^{an/2}$, which is summable, so $f(X_n) > an$ eventually a.s. □

Our next goal is to bound the time spent in vertex sets, in particular, islands.

Lemma 27 *Let G be a graph with i-anchored expansion, let S be a vertex set, and suppose $i' \leq i$ is a constant so that S is contained in $G_{i'}$. Let n be an integer, $x \in V \setminus A_i$ be a vertex with $\mathrm{dist}_i(x,S) \geq n+1$, and let T be the time the random walk on G spends in S. Then we have*

$$\begin{aligned} \mathbf{P}_x(T > 0) &\leq 2w_x^{-\frac{1}{2}}\,(1-i^2)^{\frac{n}{2}}\,i^{-2}\,|\partial S|^{\frac{1}{2}}, \\ \mathbf{E}_x T &\leq 2w_x^{-\frac{1}{2}}\,(1-i^2)^{\frac{n}{2}}\,i'^{-2}\,|S|^{\frac{1}{2}}, \\ \mathbf{E}_x T^2 &\leq 8w_x^{-\frac{1}{2}}\,(1-i^2)^{\frac{n}{2}}\,i'^{-4}\,|S|^{\frac{1}{2}}. \end{aligned}$$

For an arbitrary vertex x, these bounds hold with $n = 0$. If all edge weights are at least 1 and S is a union of islands, then we can use $i' := |S|^{-1}$.

PROOF. The quantities T, w_x, $|\partial S|$, $|S|$ do not change if they are considered (for the walk) in the graph $G_{i'}$ instead of the graph G, so we will do this.

Denote P_i, $\mathcal{G}_i$, $P_{i'}$, $\mathcal{G}_{i'}$, the Markov and Green kernels of the walks on G_i, and $G_{i'}$, respectively. Recall that $\mathcal{G}_i = \sum_{m=0}^{\infty} P_i^m$, so we have

$$\|\mathcal{G}_i\| \leq \sum_{m=0}^{\infty} \|P_i\|^m = \frac{1}{1 - \|P_i\|},$$

and so from Proposition 25 we get

$$\|P_i\| \leq (1 - i^2)^{\frac{1}{2}} \leq (1 - i^2/2), \quad \|\mathcal{G}_i\| \leq 2i^{-2}, \tag{6}$$

and these inequalities also hold with i replaced by i' everywhere. For the walk on G_i started at $y \in V \setminus A_i$, the probability of moving into S from the outside in one step is given by the function $f(y)$ which equals $\boldsymbol{\delta}_y P_{i'} \mathbf{1}_S$ outside S, and 0 in S. Thus the chance of moving into S from the outside after m steps in $V \setminus A_i$ is given by $\boldsymbol{\delta}_y P_i^m f$, and therefore

$$\mathbf{P}_x(T > 0) \leq \sum_{m=n}^{\infty} \boldsymbol{\delta}_x P_i^m f.$$

In Green kernel notation, this can be written as an inner product

$$w_x^{-1}(\mathbf{1}_x, P_i^n \mathcal{G}_i f) \leq w_x^{-1} \|\mathbf{1}_x\| \cdot \|P_i\|^n \cdot \|\mathcal{G}_i\| \cdot \|f\|.$$

The norms are all $L^2(V \setminus A_i, w)$, and the last inequality follows from the Schwarz inequality and the norm bounds. Since $f \leq 1$, the last norm is bounded above by $(f, 1)^{\frac{1}{2}}$, which equals $|\partial S|^{\frac{1}{2}}$. The first claim of the lemma now follows from (6).

For the expected value, write

$$\mathbf{E}_x T = \sum_{m=0}^{\infty} \boldsymbol{\delta}_x P_i^n P_{i'}^m \mathbf{1}_S = w_x^{-1}(\mathbf{1}_x, P_i^n \mathcal{G}_{i'} \mathbf{1}_S).$$

Since $\|\mathbf{1}_S\| = |S|^{\frac{1}{2}}$, the norm bound on the last formula and (6) give the second claim of the lemma.

Finally, denote $\{X_k\}$ the random walk on $G_{i'}$. Then

$$\mathbf{E}_x T^2 = \mathbf{E}_x \sum_{\substack{s,t > n \\ y,z \in S}} \mathbf{1}(X_s = y) \mathbf{1}(X_t = z),$$

and summing twice on or under the diagonal and extending the range of y gives the upper bound

$$2\mathbf{E}_x \sum_{\substack{s>n \\ d\geq 0}} \sum_{y\in V} \mathbf{1}(X_s = y)\mathbf{1}(X_{s+d} \in S).$$

By the Markov property this equals

$$2\sum_{\substack{s>n \\ d\geq 0}} \sum_{y\in V} \mathbf{P}_x(X_s = y)\mathbf{P}_y(X_d \in S) = 2 \cdot \boldsymbol{\delta}_x P_i^n \mathcal{G}_{i'}\mathcal{G}_{i'}\mathbf{1}_S.$$

The third claim of the lemma follows if we write this as an inner product and use norm bounds, as before. Omitting the estimates for the first n steps gives the proof for general x. Lemma 22 implies that we can use $i' := |S|^{-1}$. □

The anchored expansion property suggests that large islands cannot be very frequent. The following lemma proves such a statement from the point of view of the random walk. It uses the hypotheses and the resulting decomposition of Proposition 24.

Lemma 28 *Consider countries C whose land is visited by time N_m, and let M_m be the volume of the largest such land. Then we have*

$$\limsup \frac{M_m}{\log m(\log\log m)^2} < c < \infty \qquad a.s.$$

Proof. For a positive b, let $g(n) := b\log n(\log\log n)^2$, and let $\mathcal{A}_\ell$ be the event that the land of a country of level ℓ is visited by time $g^{-1}(2^\ell)$. It suffices to prove that only finitely many of these events happen, which will follow if $\mathbf{P}\mathcal{A}_\ell \leq 2^{-\ell}$ for every large ℓ. Consider ℓ so large that the starting point of the walk is not contained in any level ℓ country, and $r(\ell) \geq 1$. Let T_C denote the first hitting time of a country C. Let $\mathcal{A}_{\ell,C}$ denote the event that the land of the country C is visited by time $g^{-1}(2^\ell)$, and let $\mathcal{A}_C$ denote the event that the land of the country C is ever visited. The event $\mathcal{A}_{\ell,C}$ implies $\mathcal{A}_C$ and $T_C \leq g^{-1}(2^\ell)$, and therefore

$$\mathbf{P}\mathcal{A}_{\ell,C} \leq \sum_{t=1}^{g^{-1}(2^\ell)} \mathbf{P}(\mathcal{A}_C|T_C = t)\mathbf{P}(T_C = t).$$

Summing over level ℓ countries we get

$$\begin{aligned}\mathbf{P}\mathcal{A}_\ell &\le \sum_{C\in\Xi_\ell}\sum_{t=1}^{g^{-1}(2^\ell)} \mathbf{P}(\mathcal{A}_C|T_C=t)\mathbf{P}(T_C=t)\\ &\le \sup_{\substack{C\in\Xi_\ell\\ 1\le t\le g^{-1}(2^\ell)}} \mathbf{P}(\mathcal{A}_C|T_C=t)\sum_{t=1}^{g^{-1}(2^\ell)}\sum_{C\in\Xi_\ell}\mathbf{P}(T_C=t). \qquad (7)\end{aligned}$$

For fixed t, the events in the inner summand of (7) are disjoint, so the second factor is bounded above by $g^{-1}(2^\ell)$. If C is a level ℓ country with land S, then

$$\text{dist}_i(X_{T_C},S)=\lfloor r(\ell)\rfloor.$$

Therefore by the Strong Markov Property and Lemma 27, $\mathbf{P}(\mathcal{A}_C|T_C=t)$ is not more than

$$2w_x^{-\frac{1}{2}}(1-i^2)^{(r(\ell)-2)/2}i^{-2}|\partial S|^{\frac{1}{2}} \le c'\exp(-c_1'2^\ell/\ell^2)2^{\ell/2}\le c2^{-\ell}\exp(-c_12^\ell/\ell^2).$$

Then by (7), $\mathbf{P}\mathcal{A}_\ell \le g^{-1}(2^\ell)c2^{-\ell}\exp(-c_12^\ell/\ell^2)$ and it suffices to prove that

$$g^{-1}(2^\ell)\le c^{-1}\exp(c_12^\ell/\ell^2).$$

We apply g to both sides and use its monotonicity to transform the above to

$$2^\ell \le b(c+c_1\frac{2^\ell}{\ell^2}(\ell\log 2-2\log\ell)^2).$$

This certainly holds for all large ℓ if b is large. □

The following corollary will be used in a later section. It implies that from the point of view of speed, distance can be measured while walking on water.

Corollary 29 *Set*

$$H_m := \inf_{N_{m-1}<n\le N_m}|X_n|. \qquad (8)$$

Then we have $\lim(H_m/|X_{N_m}|)=1$ *a.s.*

Proof. Between times N_{m-1} and N_m the walker is on an island with diameter bounded above by the volume M_m of the largest land visited by time N_m. Thus we have $|X_{N_m}|-M_m-1\le H_m\le |X_{N_m}|$. Dividing by $|X_{N_m}|$, and using the lemma together with (5) proves the corollary. □

3.5 Lower bound on the speed

This section contains the proof of Theorem 17. We also give some counterexamples indicating why the bounded geometry condition is important.

Theorem 17 *There exists $c > 0$ so that the random walk on a weighted graph with w^*-bounded geometry satisfies* $\liminf |X_n|/n \geq c\mathbf{i}(G)^7 w^{*-3}$ *a.s.*

PROOF. Let G be a graph with the anchored expansion property and w^*-bounded geometry, and consider the construction of countries from Proposition 24. Using the notation of the previous section, we can decompose the inverse lim inf speed $\underline{S}^{-1}$ as

$$\begin{aligned} \limsup n/|X_n| &= \limsup_m \sup_{N_{m-1}<n\leq N_m} (n/|X_n|) \\ &\leq \limsup_m (N_m/H_m) = \limsup (N_m/|X_{N_m}|). \end{aligned} \tag{9}$$

H_m in the above expression is defined in (8), and the last equality follows from Corollary 29. Let $K_m := N_m - m$ denote the time spent in the islands up to time N_m. By (9) we have

$$\underline{S}^{-1} \leq \limsup (m/|X_{N_m}|)(1 + \limsup (K_m/m)).$$

The first factor in the last expression is the inverse of the lim inf speed in the graph G_i, for which we have the bound (5). Thus in order to show that $\underline{S}$ is greater than a constant a.s. it suffices to find constants b_ℓ so that

$$\limsup (K_m/m) = \limsup (K_{m^2}/m^2) \leq \sum_{\ell \geq 1} b_\ell < \infty \quad \text{a.s.} \tag{10}$$

The equality holds since K_m is non-decreasing.

For each ℓ, if $X_0 = v$ is contained in a level ℓ country C, then set $C_{\ell,0} := C$, otherwise set $C_{\ell,0} := \emptyset$. Set $\tau_{\ell,0} := 0$, and for $k \geq 1$ define

$$\tau_{\ell,k} = \min\{n \geq \tau_{\ell,k-1} + 1 : X_n \in C =: C_{\ell,k} \text{ for some } C \in \Xi_\ell \setminus \{C_{\ell,k-1}\}\}.$$

Also, for $k \geq 0$, define the time spent in the land between stopping times:

$$T_{\ell,k} = \#\{n : \tau_{\ell,k} \leq n < \tau_{\ell,k+1},\ X_n \in C_{\ell,k} \cap A_i\}.$$

We will use the rough bound $K_m \leq \sum_{\ell=1}^{\infty} \sum_{k=0}^{m} T_{\ell,k}$. Since each vertex is contained in at most finitely many countries, we have $T_{\ell,0} = 0$ for all but finitely many ℓ. So for (10) it suffices to find summable b_ℓ such that

$$\sum_{m \geq 1, \ell \geq 0} \mathbf{P}(\sum_{k=1}^{m^2} T_{\ell,k} > b_\ell m^2) < \infty. \tag{11}$$

Now fix ℓ, and suppose that $X_0 = v$ is on the inner vertex boundary of a level ℓ country with land R. If $r(\ell) \geq 1$, then this means that v is in the ocean and $\text{dist}_i(v, R) = \lfloor r(\ell) \rfloor$. Lemma 27 with $i' := |R|^{-1}$ gives

$$\mathbf{E}T_{\ell,0}^2 \leq 8(1-i^2)^{(r(\ell)-2)/2} |R|^{4.5} \leq 8(1-i^2)^{a2^{\ell-1}/\ell^2 - 1} 2^{4.5\ell} =: a_\ell^2. \tag{12}$$

If $r(\ell) < 1$, then v is contained in the land R, and this bound still holds (although it is very rough) by Lemma 27 applied to a general starting point. By the Markov property, this implies that for all $k \geq 1$, we have $\mathbf{E}(T_{\ell,k}^2 | \mathcal{F}(\tau_{\ell,k})) \leq a_\ell^2$, where $\mathcal{F}(\tau_{\ell,k})$ denotes the standard σ-field at the stopping time $\tau_{\ell,k}$, that is the σ-field generated by information available up to time $\tau_{\ell,k}$. Define

$$S_{\ell,m} := \sum_{k=1}^{m} \left(T_{\ell,k} - \mathbf{E}\left(T_{\ell,k} | \mathcal{F}(\tau_{\ell,k})\right)\right) \geq \sum_{k=1}^{m} (T_{\ell,k} - a_\ell).$$

Since $\{S_{\ell,m}\}_{m \geq 1}$ is a martingale, we can write

$$\text{Var } S_{\ell,m} = \sum_{k=1}^{m} \mathbf{E}\text{Var}\left(T_{\ell,k} | \mathcal{F}(\tau_{\ell,k})\right) \leq m a_\ell^2.$$

Therefore, if $b_\ell > a_\ell$, then Chebyshev's inequality gives

$$\mathbf{P}(\sum_{k=1}^{m} T_{\ell,k} > m b_\ell) \leq \mathbf{P}(S_{\ell,m} > m(b_\ell - a_\ell)) \leq \frac{m a_\ell^2}{m^2 (b_\ell - a_\ell)^2}.$$

Thus if we set, for example, $b_\ell := (\ell + 1) a_\ell$, then it is clear from looking at the expression of a_ℓ that the conditions of (11) and (10) are satisfied. We thus have proved that the speed is greater than a constant depending on i and w^* only.

It remains to give a bound on the constant in terms of i and w^*. Since b_ℓ can be large when ℓ is small, in order to get a reasonable bound, we need to deal with countries at or below some minimal level ℓ_0 separately.

The value of ℓ_0 will be determined later, for now just assume that $2^{-\ell_0} \leq i$. Then by Lemma 22 the land of countries of level up to ℓ_0 is contained in $V \setminus A_{2^{-\ell_0}}$. Let K_m^* denote the time the random walk spends in $V \setminus A_{2^{-\ell_0}}$ by time N_m, and let K'_m denote the time the walk spends in the land of countries of level greater than ℓ_0 by time N_m. We then have $N_m \leq K_m^* + K'_m$.

Note that the sequence $\{K_m^*/|X_{N_m}|\}$ is a subsequence of $\{m/|X_{N'_m}|\}$, where N'_m is the time of the mth visit to $G_{2^{-\ell_0}}$. By (5), the lim sup of this sequence, and thus the lim sup of the first one, is at most $(2^{\ell_0})^2 \log w^*$. The bound (5) on $\limsup(m/|X_{N_m}|)$, (9), and the bound (10) on $\limsup(K'_\ell/m)$ from the first part of the proof imply

$$\begin{aligned} \underline{S}^{-1} &\leq \limsup(K_m^*/|X_{N_m}|) + \limsup(m/|X_{N_m}|)\limsup(K'_m/m) \\ &\leq (2^{\ell_0})^2 \log w^* + i^{-2} \log w^* \sum_{\ell > \ell_0} b_\ell. \end{aligned} \tag{13}$$

We now want to choose an ℓ_0 so that the last sum is small, say each term b_ℓ is at most $2^{-\ell}$. From (12), since $\ell \geq 1$, we have

$$2^\ell b_\ell \leq \exp(c\ell - c_2 \alpha^{-1} 2^\ell/\ell^2) \tag{14}$$

with $\alpha := w^* i^{-3} \vee 2$. A simple computation shows that there is a constant $c_3 = c_3(c, c_2) \geq 1$ so that the right hand side of (14) is at most 1 if $\ell \geq \ell_0$, where ℓ_0 is chosen so that

$$2^{\ell_0} = c_3 \alpha (\log_2 \alpha)^3.$$

Using this choice of ℓ_0, from (13) we conclude that

$$\underline{S}^{-1} \leq c_3^2 \alpha^2 (\log_2 \alpha)^6 \log w^* + i^{-2} \log w^* \leq c^{-1} w^{*3} \mathbf{i}(G)^{-7}. \quad \square$$

Example 30 Consider the binary tree with edge weights 1, and for each n attach an extra vertex to each vertex at distance n from the root by an edge with weight $1/(n \log n)$. Add a self-loop to each new vertex so that it will have weight 1. This graph clearly has anchored expansion. The walk will visit infinitely many of these new vertices by the Borel-Cantelli Lemma, and at each visit it has at least constant probability to spend time at least $n \log n$ at the vertex. Thus in this graph the speed is zero; this shows that in Theorem 17 the bounded geometry condition cannot be left out, nor replaced by bounds on the vertex weights.

Example 31 It follows from the bounded geometry property that positive transition probabilities are bounded below. This is another weaker condition, but too weak for Theorem 17. Define the **pipe** of length n as the nearest neighbor graph on $0, 1, 2, \ldots, n$. Consider the binary tree, and for every n and vertex at distance n, add a pipe of length $2k$ with edge weights $1, 2^{-1}, \ldots, 2^{-k+1}$, 2^{-k}, $2^{-k+1}, \ldots, 1$ with $2^k \approx n \log n$. The argument of the previous example applies again.

3.6 A heat kernel bound

This section contains the proof of Theorem 18 and examples showing that the bound there is sharp up to the constant factor in the exponent.

Theorem 18 *Let G be a weighted graph with the anchored expansion property and w^*-bounded geometry. Let $\alpha := \mathbf{i}(G)^2(w^{*2}/2)^{-1/3}/9$. For every vertex x there is an N so that*

$$p^n(x, y) < e^{-\alpha n^{1/3}} \quad \textit{for all } n > N,\ y \in V.$$

Proof. Fix a vertex x, and let $a_n := an^{1/3}$ for $a > 0$ to be determined later. Let $i := \frac{2}{3}\mathbf{i}(G)$, let $A_{i,n} \subset A_i$ be the union of islands with volume at least a_n, and define the **territory** of such an island C as the set of vertices $v \in V$ with

$$\mathrm{dist}_i(v, C) \le a_n i/(4w^*). \tag{15}$$

Lemma 23 implies that there are only finitely many n for which x is contained in the territory of an island of $A_{i,n}$. Consider large n for which *(i)* this does not happen and *(ii)* the right hand side of (15) is at least 1. Condition *(ii)* and the definition of dist_i ensures that the inner vertex boundary of the territory of an island is a subset of the ocean, $V \setminus A_i$. Condition *(i)* implies that $1/a_n < i$, and, as shown in Section 3.3, $A_{1/a_n} \subset A_i$. By Lemma 22, islands of A_i with volume less than a_n do not intersect A_{1/a_n}, so $A_{1/a_n} \subset A_{i,n}$, and we have $x \in V(G_{1/a_n})$.

Let $A'_{i,n}$ denote the union of islands of $A_{i,n}$ which are at distance at most n from x, and let p'^n denote the transition kernel of the walk on G_{1/a_n}. Note that $p^n(x, y)$ is the sum of the probabilities of paths of length n starting at x and ending at y. Each such path stays in G_{1/a_n} or visits $A'_{i,n}$. The total probability of the first kind of paths is at most $p'^n(x,y)$ (regarded as 0 if $y \in A_{1/a_n}$), so for all y we have

$$p^n(x,y) \le p'^n(x,y) + \mathbf{P}_x(\{X_k\} \text{ hits } A'_{i,n}).$$

The first term on the right satisfies

$$\begin{aligned} p'^n(x,y) &= w_x^{-1}(\mathbf{1}_x, P^n_{1/a_n}\mathbf{1}_y) \le w^{*\frac{1}{2}}\|P_{1/a_n}\|^n \\ &\le w^{*\frac{1}{2}}(1-a_n^{-2})^{\frac{n}{2}} < w^{*\frac{1}{2}}\exp(-\tfrac{1}{2}na_n^{-2}). \end{aligned} \tag{16}$$

The first inequality follows from Cauchy-Schwarz, the second from Proposition 25 and the fact that G_{1/a_n} has Cheeger constant at least $1/a_n$.

Suppose that there is a union R_n of $n+1$ islands in $A'_{i,n}$ so that the territory of the first intersects the territory of all the others. Then there is bridge structure B interconnecting $R_n \cup \{x\}$ with

$$\#(B \cup \{x\} \setminus A_i) \le n + n(\tfrac{i}{2}w^{*-1}a_n - 1) \le w^{*-1}\tfrac{i}{2}|R|.$$

In the second expression, second term in parentheses is an upper bound on the number of vertices in a bridge connecting two islands with intersecting territories, and the first n is an upper bound for the number of vertices needed for the connection to x. Lemma 23 then implies that there are finitely many n such that R_n exists. Thus for all large n, it is possible to n-color islands in $A'_{i,n}$ so that islands of the same color have disjoint territories. For such n, the probability of hitting some island is bounded by n times the maximal probability of hitting some island of a given color.

Suppose that the walk starts at a vertex v on the inner vertex boundary of the territory of an island $C \subset A'_{i,n}$. By construction, this means that

$$\text{dist}_i(v,C) = \lfloor a_n i/(4w^*)\rfloor \ge 1.$$

Also note that another application of Lemma 23 shows that for some c and all large n, $A'_{i,n}$ cannot contain islands with volume larger than cn. For such n, by Lemma 27 the probability of hitting C is bounded by

$$2w_x^{-\frac{1}{2}}(1-i^2)^{(a_n i/(4w^*)-2)/2}i^{-2}|\partial C|^{\frac{1}{2}} \le c(i,w^*)n^{1/2}(1-i^2)^{(a_n i/(4w^*)-2)/2}.$$

In the first n steps the walker has at most n occasions to be at the inner vertex boundary of some island of a given color. Thus by the Markov property we get the bound

$$\mathbf{P}_x(\{X_k\} \text{ hits } A'_{i,n}) \leq n \cdot n \cdot cn^{1/2} \exp\left(\log(1-i^2) i a_n/(8w^*)\right). \tag{17}$$

There exists an $a < (2\alpha)^{-1/2}$ so that the exponents of (16) and (17) are at most $-cn^{-1/3}$ with $c > \alpha$. The statement of the theorem follows. □

Example 32 Let $\{X_n\}$ be the nearest neighbor walk on the nonnegative integers started at 0. There is a constant a so that $\mathbf{P}(X_1, \ldots, X_{n^3} < n) > e^{-an}$ for all n. Consider the binary tree with pipes of length ℓ_n attached to a vertex v_n at distance ℓ_n from the root o for some rapidly increasing sequence $\{\ell_n\}$. This graph has anchored expansion. However, consider the set of possible paths of length ℓ_n^3 which start and end at o. A subset of these start at o, travel on a shortest path to the opposite end of the pipe starting at v_n, spend time $\ell_n^3 - 4\ell_n$ in the pipe, and use the remaining time to return to o. By the above, the probability measure of this set of paths is at least $(1/4)^{4\ell_n} e^{-a\ell_n}$, and we get $p^{\ell_n^3}(o, o) > e^{-c\ell_n}$. This shows that the conclusion of Theorem 18 is sharp up to the constant in the exponent.

Example 33 Chen and Peres (1998) showed that a supercritical Galton-Watson tree has anchored expansion, so the above theorem gives the $e^{-cn^{1/3}}$ heat kernel upper bound. In this case, such bounds are immediate from results of Piau. For Galton-Watson trees where the probability of non-branching (zero or one offspring) is positive this bound is easily seen to be sharp up to the constant in the exponent (see Piau 1998).

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