Stat 2211 Assignment 1

Due Tuesday, February 3, beginning of class

The last two problems are easier.

Problem 1. Consider a subgraph of the Euclidean lattice $\mathbb{Z}^d$ with vertex set $V = \{0, \dots n\}^d$, and edges between points that have L^1-distance 1. Let $a = (0, \dots, 0)$ and $z = (n, \dots, n)$. Show that as $n \to \infty$, the effective resistance satisfies

$$R_{az} \asymp \begin{cases} n, & d = 1 \\ \log n, & d = 2 \\ 1, & d \geq 3 \end{cases}$$

(Hint: for the lower bound on resistance, merge vertices. For the upper bound, construct a flow.) Recall that $\asymp$ means that the two sides are bounded between fixed constant times one another.

Problem 2. Let g be a function on a finite connected graph that is harmonic at all points except perhaps a singe point a. Show that g is a constant function.

Problem 3. (Is random walk in dimensions 2.5 recurrent?) Let $r_1, r_2, \dots$ be a positive nondecreasing sequence, and consider the set

$$\{(x, y, z) \in \mathbb{Z}^3 \, : x \geq 1, |y| \leq r_x, |z| \leq r_x\}.$$

with edges between points of L^1-distance 1.

(a) Show that this graph is transient if and only if

$$\sum_{x=1}^{\infty} \frac{1}{r_x^2} < \infty$$

(Hint: for the lower bound on resistance, merge vertices. For the upper bound, construct a flow.)

(b) Let V_n denote the number of vertices at distance at most n in the graph from $(1, 0, 0)$, and assume that

$$d = \lim_{n \to \infty} \frac{\log V_n}{\log n}$$

exists; call this the dimension of the graph. For what dimensions can the graph be transient? For what dimensions can it be recurrent?

Problem 4. Let G be a finite connected graph, and let a,z be two vertices.

(a) Show that there exists a flow of strength 1 of minimal energy (hint: show that the space of such flows is compact and use that the energy is a continuous function).

(b) Show that there is a unique flow of minimal energy (use strict convexity of energy).

(c) Perform these steps for infinite graphs with $z = \infty$. More precisely, for flows with source a but no sink.

Problem 5. Let $G = (V, E)$ be a finite graph, and consider the usual inner product

$$\langle f, g \rangle = \sum_{v \in V} f(v) g(v)$$

for real-valued functions f, g from the set of vertices of G. Also, consider the usual inner product

$$\langle p, q \rangle = \sum_{e \in E} p(e) q(e)$$

for real-valued functions from the set of edges E.

(a) Prove the following discrete analogue of the integration-by-parts formula. Let f be a vertex function and q be an edge function. Then (with the divergence and gradient notation)

$$\langle f, \nabla \cdot q \rangle = \langle \nabla f, q \rangle.$$

What does this have to do with integration by parts?

(b) Recall that the energy of a flow is given by $\langle I/c, I \rangle$, where $I/c = \nabla V$. Use part (a) to show that the voltage difference at a and z of the unit current flow is equal to its energy.

Problem 6. Let $G = (V, E)$, $V = \{0, 1, \ldots, n-1\}$ and edges between x, y if $x - y = \pm 1 \mod n$, a cycle. Start a random walk at 0 and let T be the time of hitting the vertex a. Compute $\mathbf{E}T$ (hint: this is not a commute time, but...).

Problem 7. Let $\mathbb{T}_d$ be a connected graph where each vertex has degree d and there are no cycles (there is only one such graph, it is called the d-regular infinite tree). Start a random walk at an arbitrary vertex v, and compute the total expected number of returns to v.