Stat 2211 Final exam
April 16. 2009

Your name
Student ID
email

There are 9 problems on this exam. Do only 8 of the 9 problems, and cross out the remaining problem. (If you don't cross anything out, Problems 1-8 will be graded). Each problem is worth 10 points. This is a closed book exam. The problems are not of the same difficulty, and it helps to start with the easier problems.

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
9	
Total	

Problem 1. Let X_i be i.i.d. Uniform$[0, 1]$ random variables, and let $S_n = X_1 + \ldots + X_n$. Find $c_1, c_2 > 0$ so that $P(S_n < n/3) < c_1 \exp(-c_2 n)$ for all n. Prove your claims.

Problem 2. Let Y_n be a branching process with offspring distribution with mean 1 and positive variance. Prove (without using any theorems on branching processes) that Y_n dies out with probability 1.

Problem 3. Show that there exists $c, \beta > 0$ so that for any connected graph on n vertices with at least one cycle the mixing time for the random walk is at most cn^{β}.

Problem 4. Show that every irreducible recurrent Markov chain on a countable state space has at least one stationary measure.

Problem 5. Show (not using any theorems for Markov chains) that if x is a recurrent state of Markov chain and $K^n(x, y) > 0$ for some y, then y is also a recurrent state.

Problem 6. Find a nonnegative martingale X_n so that $EX_n = 1$ for all n, but X_n a.s. converges to a random variable X with $EX \neq 1$ and $\mathrm{Var}(X) > 0$.

Problem 7. Fix $b < 1$. A finite, connected graph G is called a b-expander if the eigenvalues of the random walk of G are all at most b in absolute value, except for the largest one (which equals 1).

Show that there exists a constant c depending on b and only so that for every b-expander $G = (V, E)$ with vertex degrees at most 4, for $|V| > 1$, for every $A \subset V$ satisfying $\pi(A) > 1/2$ and every $x \in V$ the simple random walk on the graph started at x satisfies

$$P_x(X_{c \log |V|} \in A) > 1/3.$$

Problem 8. Let M_n be a martingale with mean zero and increments at most 1. Show that for every $\alpha > 1/2$ we have $M_n/n^\alpha \to 0$ almost surely.

Problem 9. Let X, Y be mean zero, bounded, independent random variables, and let $Z = X + Y$. State the definition of conditional expectation, and use the definitions carefully to show that $E[Z|\sigma(X)] = X$. In this problem, you have to take extra care that all the steps are done rigorously.