# Random walks on finite convex sets of lattice points

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**Abstract**

This paper examines the convergence of nearest-neighbor random walks on convex subsets of the lattice $\mathbb{Z}^d$. The main result shows that for fixed $d$, $O(\gamma^2)$ steps are sufficient for a walk to "get random", where $\gamma$ is the diameter of the set. Toward this end a a new definition of convexity is introduced for subsets of lattices, which has many important properties of the concept of convexity in Euclidean spaces.

Keywords: convexity, random walks, convergence rate, lattices

# Introduction

Consider a finite subset $V$ of the $d$-dimensional lattice $\mathbb{Z}^d$ equipped with the natural nearest neighbor graph structure: two vertices are connected if their Euclidean distance is 1. One can define a random walk on this graph: in each step we pick one of the $2d$ directions at random, and if there is another vertex in that direction, we go there. This random walk has the Markov kernel:

$$K(x,y) = \begin{cases} 1/2d & \text{for } x \neq y \text{ neighboring points} \\ 1 - n(x)/2d & \text{for } x = y \end{cases} \tag{1}$$

where $n(x)$ is the number of neighbors of $x$ in the graph $V$. This walk has *uniform* stationary distribution $\pi_x \equiv 1/k$, where $k$ is the number of vertices in $V$. We are interested in how long it takes get close to this stationary distribution.

The main theorem in this paper provides a solution to an open problem stated in [1], claiming that $c_d\gamma^2$ steps are sufficient, where $\gamma$ is the graph-theoretic diameter of $V$, and $c_d$ is a constant which depends only on the dimension $d$. The result was proved by Diaconis and Saloff-Coste for d=2, and was stated open by the same authors for higher dimensions. The proof of this result uses geometric arguments and the techniques developed in [1].

The first chapter introduces a new notion of convexity for sets $V$ of lattice points. Then follows a proof that convex sets contain geodesic paths which are close to straight in the Euclidean sense. Chapter 2 reviews geometric techniques used to estimate the second-largest eigenvalue and to get convergence bounds, and proves the main theorem.

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# Chapter 1. Convexity

## 1 The definition

The goal of this section is to establish a definition of convexity on sets of lattice points. Based on the notion of convexity in $\mathbb{R}^d$, we would like a convex set $V \subset \mathbb{Z}^d$: (1) to be connected in the natural graph structure, (2) not to have any 'bottlenecks', and equivalently (3) to contain a path between any two of its points which is close to 'straight' in the Euclidean sense.

As sets of lattice points are naturally equipped with a graph structure, one could define convexity borrowing the definition form graph theory. A convex subgraph $G'$ of a graph $G$ must contain all shortest paths between its vertices. Thus a set $V \subset \mathbb{Z}^d$ can be defined convex if its natural graph is a convex subgraph of $\mathbb{Z}^d$. Unfortunately, under this definition the only convex sets of lattice points are boxes, so it is too restrictive.

One can try to define $V$ to be convex if it is the set of lattice points inside a convex set in $\mathbb{R}^d$. This definition violates the condition of connectedness even when $d = 2$. There are 'convex' sets, such as $V = \{(z, z) | 0 \leq z \leq n\}$, whose natural graph is not connected.

To correct this, we can define convex sets $V \subset \mathbb{Z}^d$ to be sets of lattice points inside a convex set in $\mathbb{R}^d$, if they define a connected graph. This way we get a definition that works well in the case $d = 2$. Diaconis and Saloff-Coste use this definition in [1] to prove a two dimensional version of the main theorem of this paper. However, the same definition will not work well in 3 dimensions. A counterexample named 'dog' is given in [1]: Consider the set $Dog_n = \{(x, y, 0) | 0 \leq x, y \leq n\} \cup \{(x, y, 1) | -n \leq x, y \leq 0\}$ (see Figure 1). Note that the convex hull of $Dog_n$ contains no extra lattice points. Such a set will inevitably have a 'bottleneck' (it consists of two disjoint 'squares' connected by one edge). Also, any path between the vertices $(0, n, 0)$ and $(-n, 0, -1)$ must contain the vertex at the origin—therefore it cannot be 'straight' in the Euclidean sense.

The definition of convexity presented next will not allow such counterexamples, yet it is intuitive.

**Definition 1.1** *A set of lattice points* $V \subset \mathbb{Z}^d$ *is convex if there is a nonempty convex set* $C \subset \mathbb{R}^d$ *such that* $z \in V \Leftrightarrow d_\infty(z, C) \leq \frac{1}{2}$.

Here $d_\infty(x, y) = \max(|x_i - y_i|)$. We call the convex set $C$ the **base** for the set of lattice points $V$. There are two equivalent versions that help understand the definition. Let $\mathcal{D}$ denote the closed $d_\infty$ ball of radius $\frac{1}{2}$, which is a hypercube of volume 1. Then:

- $V$ is the set of lattice points inside $C + \mathcal{D}$, where $+$ denotes element-wise addition.

- One can think that each lattice point $z$ has its own cubicle, $z + \mathcal{D}$. These cubicles are almost disjoint (up to lower dimensional components), and they cover $\mathbb{R}^d$. Then $V$ is the set of lattice points whose cubicles intersect the convex set $C$ (see Figure 2).

The following sections will show that convex sets of lattice points are connected and have geodesic paths that are close to straight in the Euclidean sense. This means that our definition satisfies the intuitive criteria for convexity given above. It also shows that the previous counterexample sets $Dog_n$ and $V = \{(z,z) | 0 \leq z \leq n\}$ are not convex. To illustrate the strength of this definition, section 3 shows that in most cases the convex set of lattice points gained by letting $C$ be an Euclidean line is a single infinite path.

We conclude this section with the most important notation. In this paper distances are $d_\infty$, unless stated otherwise. $D(r)$ will denote the closed $d_\infty$-ball of radius $r$ about the origin, and $\mathcal{D}$ denotes $D(\frac{1}{2})$, which has volume 1:

$$D(r) := \{x \in \mathbb{R}^d | d_\infty(x, O) \leq r\} \qquad \mathcal{D} := D(\tfrac{1}{2}). \tag{2}$$

Addition and scalar multiplication in connection with subsets of $\mathbb{R}^d$ will be understood as element-wise operations. For a lattice point $z \in \mathbb{Z}^d$ we will call the set $z + \mathcal{D} := \{z\} + \mathcal{D}$ the **cubicle** of $z$. Note that the collection of all cubicles covers the entire $\mathbb{R}^d$. For two points $x, y$, we will distinguish the line $\overline{xy}$ from the line segment $xy$.

## 2 Alternative definitions

The following technical lemma provides alternative formulations of the definition of convexity. It will be used in later sections.

**Lemma 2.1** *For a finite subset $V$ of the lattice $\mathbb{Z}^d$ the following are equivalent:*

1. *$V$ is convex.*

2. *There is a closed convex set $C'$ in $\mathbb{R}^d$ and an $\varepsilon > 0$ such that: (1) $V$ is the set of lattice points at most $\frac{1}{2}$ away from $C'$. (2) there are no other lattice points in the $\frac{1}{2} + \varepsilon$ neighborhood of $C'$*

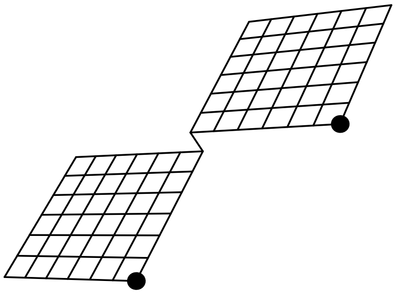

Figure 1: A small dog

*3. There is a closed convex set $C''$ in $\mathbb{R}^d$ and an $\varepsilon > 0$ such that: (1) $V$ is the set of lattice points at most $\frac{1}{2} - \varepsilon$ away from $C''$. (2) there are no other lattice points in the $\frac{1}{2}$ neighborhood of $C''$*

**Proof.**

($1 \Rightarrow 2$) For each $p \in V$, let $c_p$ be a point in the intersection of $p + \mathcal{D}$ and $C$ ($C$ as in the definition of convexity (1.1)). Let $C'$ be the convex hull of all $c_p$-s. Let $V'$ be the set of lattice points with $d_\infty$-distance at most $\frac{1}{2}$ from $C'$. Then, by construction of $C'$ we have $V \subset V'$. Also the convex hull $C'$ of the $c_p$-s is contained in $C$, because $C$ is convex, so $V' \subset V$. Since $C'$ is closed and bounded, there is a lattice point outside $V$ with minimal $d_\infty$-distance $\delta$ from $C'$. Let $\varepsilon = (\delta - \frac{1}{2})/2$. This will work.

($2 \Rightarrow 3$) Take $C''$ to be the set of points with $d_\infty$ distance at most $\varepsilon$ from $C'$.

($3 \Rightarrow 1$) Clear. □

## 3 Euclidean lines and paths in $\mathbb{Z}^d$

This section shows that for most Euclidean lines $C$, the convex set of lattice points induced by $C$ as a base is an infinite path. Let $E_n$ be the set of points in $\mathbb{R}^d$ for which at least $n$ coordinates are in $\mathbb{Z}+\frac{1}{2}$, the set of reals of the form integer$+\frac{1}{2}$. Geometrically, $E_n$ is the union of all $(d-n)$ and lower dimensional components of the cubicles $z + \mathcal{D}$ for $z \in \mathbb{Z}^d$. As an example, in 3 dimensions $E_2$ is the union of all edges. We call a line $\overline{v}$ a **proper line** if it does not intersect the set $E_2$. Informally, most lines are proper, and Lemma 4.2 shows that there is a proper line 'close' to any line. Define the **infinite path** to be the natural graph of $\mathbb{Z}$.

**Proposition 3.1** *Let $C$ be a proper line, and let $V$ be the set of lattice points whose cubicles intersect $C$. Then the infinite path and the natural graph of $V$ are isomorphic as graphs. Moreover, under this isomorphism, any section $(j, j+1, \ldots k)$ of the infinite path maps to a geodesic path of $\mathbb{Z}^d$.*

The proof of this result requires some lemmas concerning proper lines. Let $\mathcal{D}^o$ denote the interior of the hypercube $\mathcal{D}$.

**Lemma 3.2** *A proper line $\overline{v}$ is not contained in any of the hyperplanes $\xi_i = z$, where $z \in \mathbb{Z}+\frac{1}{2}$.*

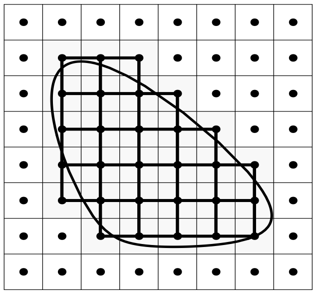

Figure 2: A convex set of lattice points and its natural graph

**Proof.** WLOG suppose that $\overline{v}$ is contained in the hyperplane $\xi_1 = \frac{1}{2}$. Then for some $a, b \in \mathbb{R}^d$ we can write our line as the set $\overline{v} = \{a + \lambda b | \lambda \in \mathbb{R}\}$, where $a_1 = \frac{1}{2}$, $b_1 = 0$, and some coordinate $b_i$ is nonzero. Therefore if we take $\lambda = \frac{1}{2} - a_i/b_i$, we get a point on $\overline{v}$ which has two coordinates equal $\frac{1}{2}$, a contradiction. $\square$

**Lemma 3.3** *Suppose that a proper line $\overline{v}$ intersects a cubicle $z + \mathcal{D}$ in a nonempty set $I$. Then $I$ is a closed interval in $\overline{v}$, containing at least two points, and at least one point from $z + \mathcal{D}^o$.*

**Proof.** Let $v$ be a unit vector parallel to $\overline{v}$. Since both $\overline{v}$ and $z + \mathcal{D}$ are closed and convex, their intersection must be a closed and convex subset of $\overline{v}$, and so a closed interval. Note that the cubicle $z + \mathcal{D}$ is determined by $2d$ inequalities of the form $\pm\xi_i \leq \pm z_i + \frac{1}{2}$. Let $p \in I$, then $p$ has at most one coordinate of the form integer$+\frac{1}{2}$. Thus there is an $\varepsilon$-ball about $p$ which satisfies all inequalities for the cubicle except for one. Then one of $p \pm \frac{\varepsilon}{2} v$ satisfies that inequality as well, call this point $p'$. $p'$ must be in the interior of $z + \mathcal{D}$, since otherwise both $p$ and $p'$ were contained in the same plane $\xi_i = z_i \pm \frac{1}{2}$. By Lemma 3.2 this would contradict the fact that $v$ is a proper line. $\square$

**Corollary 3.4** *A proper line $\overline{v}$ intersects a cubicle $z + \mathcal{D}$ if and only if it intersects its interior $z + \mathcal{D}^o$. Also, $\overline{v}$ intersects $z + \mathcal{D}^o$ in an open interval.*

**Lemma 3.5** *Let $z \neq w \in \mathbb{Z}^d$, and suppose that the cubicles $z + \mathcal{D}$ and $w + \mathcal{D}$ intersect in a point $p \notin E_2$. Then $z$ and $w$ are connected by an edge in the natural graph of $\mathbb{Z}^d$.*

**Proof.** Note that the $d_\infty$ distance of $z$ and $w$ is at most 1. Therefore the difference $|z_i - w_i|$ is at most 1, and if the two vertices are not connected, they differ in at least two coordinates $i, j$. Consider the hyperplanes $\xi_i = (z_i - w_i)/2$ and $\xi_j = (z_j + w_j)/2$. Both separate the hypercubes $w + \mathcal{D}$ and $z + \mathcal{D}$, so the common point $p$ must be contained in both planes. This means that $p$ has two coordinates of the form integer$+\frac{1}{2}$, so $p \in E_2$, a contradiction. $\square$

**Proof of Proposition 3.1. Isomorphism part.** For any distinct $z, z' \in V$, the sets $z + \mathcal{D}^o$ and $z' + \mathcal{D}^o$ are disjoint, and they intersect $C$ in disjoint nonempty open intervals (Corollary 3.4). For each $z \in V$, pick a point $p(z)$ in $(z + \mathcal{D}^o) \cap C$. Then the points $p(z)$ form a discrete set on the line $C$. We can define an order on them in one direction along the line $C$, and label them with integers so that $p_i < p_{i+1}$ for all $i \in \mathbb{Z}$. This defines a similar order on the corresponding $z$-s, which is independent of the choice of $p(z)$-s. We claim that the map $i \mapsto z_i$ is a graph isomorphism. We have to prove that $(z_i, z_j)$ is an edge if and only if $i = j \pm 1$.

(If) Consider two consecutive points $p_i, p_{i+1}$, and the corresponding $z_i, z_{i+1}$. There is no cubicle $z_j + \mathcal{D}$ that intersects the line between $p_i$ and $p_{i+1}$, since then we would have $i < j < i+1$ by our ordering. Also, the cubicles $z_i + \mathcal{D}$ cover the line $C$. Therefore $z_i + \mathcal{D}, z_{i+1} + \mathcal{D}$ cover the line segment $p_i p_{i+1}$, and since they are closed, they must intersect on the line segment (otherwise they would provide a separation of a connected segment). This means that $z_i + \mathcal{D}$ and $z_{i+1} + \mathcal{D}$ have a common point not in $E_2$, so by Lemma 3.5 $z_i$ and $z_{i+1}$ are connected by an edge in the natural graph of $V$.

(Only if) Let $(z_i, z_j)$ be an edge in $V$, then $z_i$ and $z_j$ differ in one coordinate by 1. Therefore the set $(z_i + \mathcal{D}) \cup (z_j + \mathcal{D})$ is convex, and so it contains the entire line segment $p_i p_j$. Note that the set $(z_i + \mathcal{D}) \cup (z_j + \mathcal{D})$ is disjoint from the interior of the cubicle of any third lattice point. Therefore no points on the line segment $p_i p_j$ can be in the interior of a third cubicle. In particular, there is no other point $p_k$ on this segment. This means that in our labeling $p_i$ and $p_j$ had to be consecutive points, so $i = j \pm 1$.

**Proof that the sections are geodesic.** Suppose that a section $z_j, z_{j+1}, ..., z_k$ is not geodesic, then it contains edges $(z_a, z_{a+1})$ and $(z_b, z_{b+1})$ for $a < b$, such that for one of the coordinates (WLOG the first one) we have:

$$(z_a)_1 = (z_{b+1})_1, \quad \text{and} \quad (z_{a+1})_1 = (z_b)_1 = (z_a)_1 \pm 1 \qquad (\text{WLOG} = (z_a)_1 + 1).$$

This means that the cubicles $z_a + \mathcal{D}^o$ and $z_{b+1} + \mathcal{D}^o$ are strictly separated from $z_b + \mathcal{D}^o$ by the hyperplane $\xi_1 = (z_a)_1 + \frac{1}{2}$. Therefore the line $C$, which contains the points $p_a, p_b, p_{b+1}$ in this order, must intersect that hyperplane twice (once between $p_a$ and $p_b$, and once between $p_b$ and $p_{b+1}$). Thus $C$ is contained in the hyperplane, so it cannot be a proper line, a contradiction. □

**Corollary 3.6** *(of the proof.) Let $x, y \in \mathbb{Z}^d$, and $x', y'$ be points in the interior of their cubicles respectively. Assume further that the line $\overline{x'y'}$ is proper. Then there is a geodesic path $\gamma_{x,y} = (x = z_1, z_2, ..., z_j = y)$ such that the cubicles $z_i$ intersect the line segment $x'y'$.*

## 4 Straight paths

Corollary 3.6 showed that a segment of a proper line defines a geodesic path in $\mathbb{Z}^d$. This section uses this result to prove that there are almost straight geodesic paths in any convex set of lattice points $V$.

**Proposition 4.1** *In a convex set of lattice points $V$ any two vertices $x, y$ are connected by a path $\gamma_{x,y}$ in $V$ with the following properties: (1) $\gamma_{x,y}$ is geodesic as a path in $\mathbb{Z}^d$ (2) each vertex in $\gamma_{x,y}$ has $d_\infty$ distance less than 1 from the Euclidean line $\overline{xy}$.*

Given two lattice points $x, y \in V$, our strategy is to find a nearby $x'$ and $y'$ in the base $C$ of $V$ such that the segment $x'y'$ is proper, and so the path that it defines as a base will work. We cannot do this directly, because an arbitrary base for $V$ does not always contain a proper line segment. The proof therefore requires a special base for $V$ provided by Lemma 2.1. The following lemma will also be needed:

**Lemma 4.2** *Let $x, y$ be points in $\mathbb{R}^d$, and let $\varepsilon > 0$. Then there are points $x', y'$ in the $\varepsilon$-neighborhood of $x, y$ respectively, such that the line $\overline{x'y'}$ is proper.*

**Proof.** Pick $x' \in x + D(\varepsilon)$ ($D$ being the $d_\infty$-ball defined in 2) to have rational coordinates which are not in $\mathbb{Z} + \frac{1}{2}$. Then pick $z$ in the $\varepsilon$-neighborhood of $y - x'$ such that the ratios $z_i/z_j$ are all irrational. This means that there are no points with two rational coordinates on the line $\overline{Oz}$ except for the origin. If we set $y' = z + x'$, then there will be no points with two rational coordinates except for $x'$ on the line $\overline{x'y'}$. So the line $\overline{x'y'}$ can only intersect $E_2$ in $x'$. But $x' \notin E_2$, so $x'y'$ is a proper line. □

**Proof of Proposition 4.1.** By Lemma 2.1 there is a convex set $C$ and an $\varepsilon > 0$ such that $V$ is the set of lattice points inside $C + D(\frac{1}{2})$ and there are no extra lattice points inside $C + D(\frac{1}{2} + \varepsilon)$ ($D$ being the hypercube defined in (2)). This means that $C$ and $C + D(\frac{1}{2} + \varepsilon)$ are both base sets for $V$.

Let $x, y$ be lattice points in $V$. Then $x$ is inside $C + D(\frac{1}{2}) = C + D(\frac{\varepsilon}{2}) + D(\frac{1}{2} - \frac{\varepsilon}{2})$. Therefore there is an $x^0 \in C + D(\frac{\varepsilon}{2})$ such that $d_\infty(x, x^0) \leq \frac{1}{2} - \frac{\varepsilon}{2}$. We can pick $y^0$ in a similar fashion. By Lemma 4.2 there are points $x', y'$ in the $\frac{\varepsilon}{2}$ neighborhood of $x^0$ and $y^0$ such that the line $\overline{x'y'}$ is proper.

We have picked $x'$ and $y'$ so that (1) the points $x', y'$, are contained in the base $C + D(\varepsilon)$ for $V$, and (2) $d_\infty(x, x') < \frac{1}{2}$, so $x'$ is contained in the interior of the cubicle $x + D(\frac{1}{2})$, and the same holds for the $y$-s. By convexity, $C + D(\varepsilon)$ contains the line segment $x'y'$. This means that Corollary 3.6 applies, and there is a geodesic path $\gamma_{x,y}$ in $\mathbb{Z}^d$, so that for each $z \in \gamma_{x,y}$ the cubicle $z + \mathcal{D}^o$ intersects the segment $x'y'$ and therefore the base $C + D(\varepsilon)$. Thus $\gamma_{x,y}$ is a path in $V$.

The endpoints of the segments $xy$ and $x'y'$ have $d_\infty$-distance less than $\frac{1}{2}$. It follows that any point in $x'y'$ is less than $\frac{1}{2}$ away from the line $\overline{xy}$, and so for any vertex $z \in \gamma_{x,y}$, $d_\infty(z, \overline{xy}) < 1$. □

# Chapter 2. Convergence bounds

The goal of this chapter is to prove a bound on the convergence rate of the random walk (1) on a convex subset of lattice points. We begin with a review the geometric techniques introduced by Diaconis and Saloff-Coste, then follows a proof that a finite convex set of lattice points has the geometric properties which ensure fast convergence.

## 5 Review of geometric techniques

For the sake of simplicity we only consider random walks on graphs with uniform stationary distribution. However, the theorems introduced in this section originally do not require uniformity.

**Definition 5.1** *Consider a finite connected graph $V$. A* **basis of balls** *for $V$ is a function $B : V \times \mathbb{Z}_{\geq 0} \to \mathcal{P}(V)$, which assigns to each vertex $x$ ('center') and each nonnegative integer $r$ ('radius') a subset of $V$ ('ball'), with the following properties:*

1. *B(x,0)=$\{x\}$ for all $x$.*
2. *B(x,r) is connected.*
3. *$r_1 < r_2 \Rightarrow B(x, r_1) \subset B(x, r_2)$.*
4. *There is a $\gamma$ such that $B(x, \gamma) = V$ for all $x$. The minimal such $\gamma$ is called the diameter of the graph.*

Note that any distance on a graph defines a basis of balls $B(x,r) := \{y \in V | d(x,y) \leq r\}$. In particular, the concept *basis of balls* is a generalization of a special case of this example when $d$ is the shortest path distance. This generality is important for the purposes of this paper because the proof of the main theorem uses different ($d_\infty$) balls. Diaconis and Saloff-Coste [1] provide arguments for traditional balls, but, as they note, their arguments work for general bases of balls.

**Moderate growth** is a graph theoretic property which essentially measures the dimension of a given graph with respect to a collection of balls.

**Definition 5.2** *A graph $V$ has $(A,d)$ moderate growth if*

$$|B(x,r)| \geq \frac{|V|}{A}\left(\frac{r+1}{\gamma}\right)^d \qquad \text{for all } x \in V \text{ and integers } r = 0,...,\gamma. \tag{3}$$

that is, the volume of balls must be proportional to the $d$-th power of their radius. A thorough discussion of moderate growth can be found in [2].

**Local Poincaré inequalities** are used to measure local connectedness of the underlying graph of a given Markov Kernel. One defines 'smoothed' versions of a given function $f$, by replacing the value of $f$ by the average over balls of radius $r$:

$$f_r := \frac{1}{|B(x,r)|}\sum_{y \in B(x,r)} f(y).$$

A local Poincaré inequality bounds the distance of $f$ and $f_r$ in terms of the Dirichlet form $\mathcal{E} =< f, (I-K)f >$. A Markov kernel $K$ satisfies a local Poincaré inequality with a parameter $a$ if:

$$\|f, f_r\|^2 \leq ar^2\mathcal{E}(f,f) \tag{4}$$

for all $r = 0, 1, ..., \gamma$, all functions $f$, and all vertices $x$. For a thorough exposition of local Poincaré inequalities the reader should consult the paper by Diaconis and Saloff-Coste [1], where the following result is proved:

**Theorem 5.3** *Suppose that we have a $\kappa$-regular graph, and collection of paths $\Gamma$, so that for each $x, y$, $\Gamma$ contains a path connecting $x$ with $y$. Then the Dirichlet form $\mathcal{E}(f,f)$ of the random walk on that graph satisfies the local Poincaré inequality*

$$\|f - f_r\|_2^2 \leq \eta(r)\mathcal{E}_Q(f,f)$$

*for all functions $f$, with*

$$\eta(r) = 2\kappa \max_{e \text{ edge}} \left\{ \sum_{\substack{\gamma_{x,y} \ni e \\ x,y \in B(*,r)}} \frac{|\gamma_{x,y}|}{|B(x,r)|} \right\} \tag{5}$$

*where $x, y \in B(*,r)$ means that $x, y$ are both contained in some basis ball of 'radius' $r$.*

The **least eigenvalue** $\beta_{k-1}$ of a Markov chain describes the periodicity of the chain, which is in strong connection with cycles of odd length in the underlying graph (random walk on a bipartite graph will never get random). We quote the uniform version of the theorem of Diaconis and Stroock [3], which uses odd cycles to provide a bound on the least eigenvalue.

**Theorem 5.4** *Let $K$ be a Markov kernel of a random walk on a $\kappa$-regular graph. For each vertex $x$ let $\gamma_{x,x}$ be a cycle of odd length containing $x$. Then the least eigenvalue $\beta_{k-1}$ satisfies:*

$$\frac{1}{1+\beta_{k-1}} \leq 2\kappa \max_{e \text{ edge}} \left\{ \sum_{\gamma_{x,x} \ni e} |\gamma_{x,x}| \right\}.$$

Diaconis and Saloff-Coste [1] prove that local Poincaré inequality, moderate growth, and a bound on the least eigenvalue together imply a Nash inequality and a convergence bound:

**Theorem 5.5** *Let $K, \pi$ be a random walk on a regular graph $V$. Assume that $(K, \pi)$ has moderate growth (3) and satisfies a local Poincaré inequality (4). Assume further that the least eigenvalue $\beta_{k-1}$ satisfies $\beta_{k-1} \geq -1 + \frac{1}{a\gamma^2}$. Then*

$$\|(K_x^n/\pi) - 1\|_2 \leq a_1 e^{-m/(a\gamma^2)} \quad \textit{for} \quad n = 2a\gamma^2 + m + 1$$

*with $m \geq 0$ and $a_1 = (2e(1+d)A)^{\frac{1}{2}}\,(2+d)^{d/4}$.*

We omit the proof of this theorem, which can be found in [1] It combines the polynomial bound implied by Nash inequalities for the early phase of the walk, and the exponential eigenvalue bound for the later phase. This is the most important result needed for proving the main theorem of this paper.

We will prove that the random walk defined in (1) has moderate growth and satisfies a local Poincaré inequality. This will imply the desired convergence.

Convexity is heavily used in the following sections. Recall that if $A$ and $B$ are convex subsets of $\mathbb{R}^d$, then $A+B$ is convex. If $A$ contains the origin, and $0 \leq \lambda \leq 1$, then $\lambda A \subset A$. The following identity will prove to be a useful tool. For any finite set of lattice points $V$ and the hypercube $\mathcal{D}$ defined in Section 1:

$$\operatorname{vol}(V + \mathcal{D}) = |V|. \tag{6}$$

Let $B(x,r)$ be the set of points in $V$ of $d_\infty$-distance at most $r$ from $x$. Then $B(x,r)$ forms a basis of balls as in Definition (5.1). We will use this collection of balls throughout this chapter. In the next section we will use the volume expression 6 and set inclusion arguments in $\mathbb{R}^d$ to estimate the volume growth of our graph, that is $|B(x,r)|$ as a function of $r$.

## 6 Moderate growth

The following lemma shows that $V$ satisfies a moderate growth criterion: the number of points of $V$ inside a $d_\infty$ ball grows rapidly as $r$ increases. We will use a contraction in $\mathbb{R}^d$ and the fact that the number of lattice points in a region is bounded by the volume of a slightly larger region (see (6)). We use $\varepsilon$-arguments to get better constants (eg. compare the number of lattice points inside $D(1)$ and $D(1-\varepsilon)$).

**Lemma 6.1** *Any convex subset $V$ of the lattice $\mathbb{Z}^d$ has moderate growth*

$$|B(x,r)| \geq \frac{|V|}{A}\left(\frac{r+1}{\gamma_\infty}\right)^d \quad \textit{for all } \; x \in V \textit{ and integers } r = 0,1,\ldots,\gamma_\infty, \tag{7}$$

*where $\gamma_\infty$ is the $d_\infty$ diameter of the set $V$, and $A = 5^d$.*

**Proof.** For convenient description of contractions we will assume that $x$ is the origin. By Lemma 2.2 there is a set $C$ and $\varepsilon > 0$ such that $V$ is the set of lattice points in the $\frac{1}{2} - \varepsilon$ neighborhood of $C$, and there are no additional lattice points in the $\frac{1}{2}$-negihborhood. We can write this as $V = \mathbb{Z}^d \cap (C + D(\frac{1}{2} - \varepsilon)) = \mathbb{Z}^d \cap (C + \mathcal{D})$.

Let $C' := C + D(\frac{1}{2} - \varepsilon)$, and note that $C'$ is also convex. By the above know that $V \subset C'$. Also, $V \subset D(\gamma_\infty)$, where $D(\gamma_\infty)$ is the closed $d_\infty$-ball of radius $\gamma_\infty$. Let $r+1 \leq \gamma_\infty$, then the above two relations imply

$$\frac{r+1}{\gamma_\infty}V \subset \frac{r+1}{\gamma_\infty}(C' \cap D(\gamma_\infty)) \subset C' \cap D(r+1).$$

The second inclusion follows from the convexity of $C'$. Now let $y$ be in the RHS, then there is a $y' \in C$ with $d_\infty(y',y) \leq \frac{1}{2} - \varepsilon$. Then by the definition of convexity there is a $y'' \in V$ such that $d_\infty(y'',y') \leq \frac{1}{2}$. This means that

$$d_\infty(y'',x) \leq d_\infty(y'',y') + d_\infty(y',y) + d_\infty(y,x) \leq r + 2 - \varepsilon.$$

Since $x$ and $y''$ have integer distance, this implies $d(x,y'') \leq r+1$, and so $y'' \in B(x,r+1)$. But $d(y,y'') \leq 1 - \varepsilon$ implies $y \in B(x,r+1) + D(1) \subset B(x,r) + 4\mathcal{D}$. So we have proved

$$\frac{r+1}{\gamma_\infty}V \subset B(x,r) + 4\mathcal{D}.$$

Using this, we can write

$$\frac{r+1}{\gamma_\infty}(V + \mathcal{D}) \subset \frac{r+1}{\gamma_\infty}V + \mathcal{D} \subset B(x,r) + 5\mathcal{D} = \bigcup_{y \in B(x,r)} (y + 5\mathcal{D}).$$

The volumes of these sets are related to the numbers of lattice points they contain. The LHS has volume exactly $(\frac{r+1}{\gamma_\infty})^d|V|$, and where the volume of the RHS is bounded above by $5^d|B(x,r)|$. The inequality between these volumes implies the stated moderate growth criterion. This proof works for $r \leq \gamma_\infty - 1$, and the $r = \gamma_\infty$ case is straightforward. $\square$

The following lemma bounds the number of points in a ball of radius $2r$ in terms of the number of points in a ball of radius $r$ in $V$. It will be needed in a later section. The proof goes along the same lines as the last one.

**Lemma 6.2** $|B(x,2r)| \leq 6^d|B(x,r)|$

**Proof.** Using the notation of the last lemma, we have $B(x,2r) \subset C' \cap D(x,2r)$. Therefore $\frac{1}{2}B(x,2r) \subset \frac{1}{2}(C' \cap D(x,2r)) \subset C' \cap D(x,2r)$ by convexity. Now if $y$ is in RHS, then there is a point $y' \in V$ such that $d(y,y') < 1-\varepsilon$. This implies $y' \in B(x,r)$, and so $y \in B(x,r) + 2\mathcal{D}$. Therefore:

$$\frac{1}{2}B(x,2r) + \mathcal{D} \subset B(x,r) + 3\mathcal{D}.$$

The desired inequality follows from comparing the volumes. □

## 7 Local Poincaré inequality

Poincaré and local Poincaré inequalities are usually proved by path arguments. The main part of the following proof consists of a fairly technical argument estimating the number of almost straight geodesic paths $\gamma_{x,y}$ of bounded length containing a given edge $e$. For fixed $x$, the fact that paths are almost straight forces the number of possible $y$-s to be in a $d$-dimensional cone, which we will estimate by a prism.

**Lemma 7.1** *The random walk* (1) *on the convex set of lattice points $V$ satisfies a local Poincaré inequality*

$$\|f - f_r\|^2 \leq ar^2\mathcal{E}(f,f) \tag{8}$$

*with* $a = \frac{3}{14}d^3 42^d$

**Proof.** We will use the paths constructed in Section 4, and Theorem 5.3. Using that we have to bound

$$\eta(r) = 4d \max_{e \text{ edge}} \left\{ \sum_{\substack{\gamma_{x,y} \ni e \\ x,y \in B(*,r)}} \frac{|\gamma_{x,y}|}{|B(x,r)|} \right\}$$

by an expression of the form $ar^2$. Here $x,y \in B(*,r)$ means that $x,y$ are both contained in some basis ball of 'radius' $r$.

First we will count the number of paths containing a given edge $e$. Let $\gamma_{x,y}$ be such a path with the endpoint $y$ being the one closer to $e$, that is $d_\infty(x,e_-) \geq d_\infty(y,e_+)$. We fix $x$, and for the sake of simple notation assume that it is in the origin. Also, WLOG we can assume that the coordinates $(e_1,...,e_d)$ of $e_+$ are nonnegative, and $e_1$ is maximal. Then $y$ has coordinates so that $|y_i| \leq 2e_1$, and since our paths are geodesic, we have $0 = x_1 \leq e_1 \leq y_1 \leq r$. These imply that the first coordinate of $y$ satisfies

$$e_1 \leq y_1 \leq e_1 + r/2. \tag{9}$$

*(step 1)* We constructed our paths so that the line $\overline{xy}$ must intersect the unit $d_\infty$ ball about $e_+$, say in a point $p$, where $e_1 - 1 \leq p_1$. Then, $y$ is contained in a dilation of this ball from the origin by a factor $y_1/p_1$ (see Figure 3).

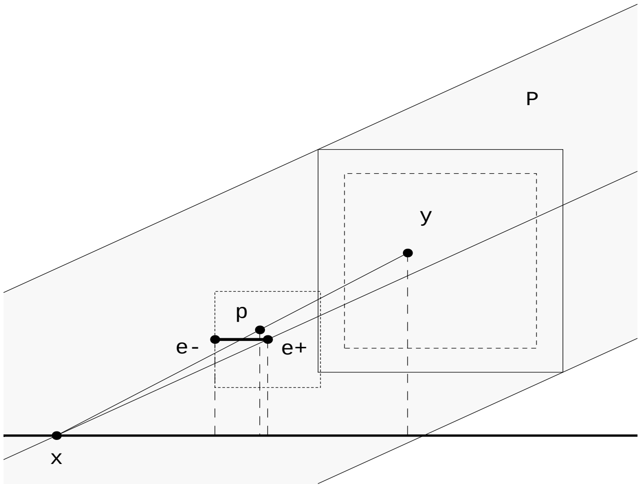

Figure 3: All possible points $y$ are contained in a *d-dimensional prism*

This means that the $d_\infty$-distance of $y$ from the line $\overline{xe_+}$ is at most $y_1/p_1 \leq 2e_1/(e_1-1)$. Thus the distance of the points in the cubicle $y+\mathcal{D}$ from $\overline{xe_+}$ is at most $2e_1/(e_1-1)+\frac{1}{2}$. For $e_1 \geq 3$, this is not more than 3.5. Therefore $y+\mathcal{D} \subset P$, where $P = \overline{xe_+} + D(3.5)$, a $d$-dimensional infinite prism, geometrically the trace of translating the hypercube $D(3.5)$ keeping its center on the line $\overline{xe_+}$.

*(step 2)* For the next part of the proof we will need the following two facts from geometry (the proofs are given later):

(10) The projection of a $d$-dimensional unit hypercube to a *(d-1)*-dimensional hyperplane has volume ("area") at most $\sqrt{d}$.

(11) Let $P$ be a $d$-dimensional prism with axis parallel to the unit vector $\vec{a}$. For any vector $\vec{v}$, let $P(\vec{v})$ be the volume of the cross section of $P$ by a *(d-1)*-dimensional hyperplane with normal vector $\vec{v}$. If $\vec{v}$ is a unit vector, then $P(\vec{a})/P(\vec{v}) = \vec{a}\cdot\vec{v}$ (a dot product).

Consider the prism $P = \overline{xe_+} + D(3.5)$. Its cross-section by a hyperplane with normal vector $x\vec{e}_+$ is a projection of the hypercube $D(3.5)$ onto that hyperplane. Thus, by (10), the volume of the cross section is at most $\sqrt{d}7^{d-1}$.

Let $\vec{a}$ be the unit vector parallel to $x\vec{e}_+$, and $\vec{\vartheta}_1$ be the first unit coordinate vector. Note that we assumed that $e_1$ is the largest coordinate of $x\vec{e}_+$. This means that the dot product $\vec{\vartheta}_1 \cdot \vec{a}$ is bounded below by $\sqrt{1/d}$. Thus by (11) the volume $P(\vec{\vartheta}_1) \leq \sqrt{d}P(\vec{a}) \leq d7^{d-1}$.

*(step 3)*

Using the result of step 1 and (9), we know that any cubicle $y+\mathcal{D}$ is contained in the intersection of the prism $P$ and the region $\{r \in \mathbb{R}^d | e_1 - \frac{1}{2} \leq z_1 \leq e_1 + \frac{r}{2} + \frac{1}{2}\}$. Any eligible candidate for $y$ occupies a cubicle of unit volume in this region. The volume of this region can be computed as height×base, that is $(\frac{r}{2}+1) \times d7^{d-1}$. Therefore the number of possible points

$y$ is at most $(\frac{r}{2}+1)d7^{d-1}$ for $e_1 \geq 3$. A simple argument shows that this bound also works for $e_1 \leq 2$. This implies that the number of paths containing $e$ is bounded above by

$$d7^{d-1}(r/2+1)\max(|B(e_-,r)|,|B(e_+,r)|) \leq d7^{d-1}(r/2+1)|B(x,2r)|$$

From Lemma 6.2 we know that $|B(x,2r)| \leq 6^d|B(x,r)|$. Since $\gamma_{x,y}$ is a geodesic path inside a ball $B(*,r)$, the length $|\gamma_{x,y}|$ is bounded by $dr$. Using this gives the bound $\eta(r) \leq \frac{1}{14}d^3 42^d r(r+2)$, which is at most $\frac{3}{14}d^3 42^d r^2$ for $r \geq 1$. □

**Proof of the facts (10) and (11).** First note the following intuitive fact (the proof can be found in [4]). Let $H(\vec{u})$ and $H(\vec{v})$ be hyperplanes with unit normal vectors $\vec{u}, \vec{v}$. Let $T$ be a polytope in $H(\vec{v})$, and let $\pi(T)$ be the orthogonal projection of $T$ to $H(\vec{u})$. Then the *(d-1)*-dimensional volumes satisfy

$$\mathrm{vol}(\pi(T))/\mathrm{vol}(T) = \vec{u}\cdot\vec{v}. \tag{12}$$

The fact (11) immediately follows from this expression when one notices that the perpendicular cross-section of an infinite prism can be thought of as an orthogonal projection of any other cross-section.

For (10), let $\mathcal{D}$ be a $d$-dimensional unit hypercube as defined in (2). Let $p$ be a plane with normal vector $\vec{v}$. WLOG we can assume that all coordinates of $\vec{v}$ are nonnegative. Now each line parallel to $\vec{v}$ that intersects the hypercube $\mathcal{D}$ must intersect one of the $d$ faces (that is, *(d-1)*-dimensional components) that are in the hyperplanes $\xi_i = \frac{1}{2}$. Therefore the projection of the hypercube is the same as the projection of the union of these faces. Each face is a *(d-1)*-dimensional hypercube of area 1. Thus by (12), the area of the projection of such a face is given by $\vec{\vartheta}_i \cdot a$, where $\vec{\vartheta}_i$ is the $i$-th coordinate vector. Thus the maximal area of the projection is given by:

$$\sup_{\vec{v}}\left\{\sum_{i=1}^{d} = \vec{\vartheta}_i \cdot \vec{v}\right\}$$

The supremum is taken over all unit vectors $\vec{v}$. Note that the sum on the RHS is just the sum of the coordinates of the unit vector $\vec{v}$, which is bounded by $\sqrt{d}$ by the inequality between the arithmetic mean and the root-mean-square. □

## 8 A bound on the least eigenvalue

The main theorem requires a bound on the least eigenvalue. For each $x$, let $\gamma_{x,x}$ be a cycle of vertices whose first coordinates are $(x_1, x_1+1, x_1+2, ..., x_1+i, x_1+i, x_1+i-1, ..., x_1)$, and the rest of the coordinates are the same as those of $x$. Here the lattice point $(x_1+i, x_2, ..., x_d)$ is on the boundary of $V$, that is it has a self-loop attached to it. In other words, the loop $\gamma_{x,x}$ goes straight to the boundary, uses the loop there and then returns to $x$. This way each edge is used in at most $\gamma_\infty$ times by the $\gamma_{x,x}$-s, since there are at most $\gamma_\infty$ vertices with the same first coordinate. Also, each cycle has length at most $2\gamma_\infty + 1 \leq 3\gamma_\infty$. Therefore Theorem 5.4 gives the bound

$$\beta_{k-1} \geq -1 + \frac{1}{4d \times 3\gamma_\infty^2} \geq -1 + \frac{1}{a\gamma_\infty^2} \tag{13}$$

where $a$ is the constant of the local Poincaré inequality (8).

## 9 Bound on the convergence rate

The moderate growth criterion (7), the local Poincaré inequality (8), and the least eigenvalue bound (13) together imply a bound on the convergence rate by Theorem 5.5. Thus we have proved the theorem:

**Theorem 9.1** *Let $V$ be a finite convex subset of the lattice $\mathbb{Z}^d$. Then the random walk (1) on $V$ satisfies the convergence bound:*

$$\|K_x^n/\pi - 1\|_2 \leq a_1 e^{-m/(a\gamma_\infty^2)} \quad \textit{for} \quad n = 2a\gamma_\infty^2 + m + 1 \tag{14}$$

*with $m \geq 0$, $a_1 = (2e(1+d)5^d)^{\frac{1}{2}}(2+d)^{d/4}$, and $a = \frac{3}{14}d^3 42^d$.*

In particular, this means that for fixed $\varepsilon > 0$, it takes at most $c\gamma^2$ steps for the distance (14) to get less then $\varepsilon$, where $\gamma$ is the graph-theoretic diameter ($\gamma \geq \gamma_\infty$). Here the constant $c$ depends only on $\varepsilon$ and the dimension $d$.

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